REPRESENTABLE K-THEORY OF
SMOOTH CROSSED PRODUCTS BY $\mathbb{R}$ AND $\mathbb{Z}$

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ABSTRACT. We show that the Thom isomorphism and the Pimsner-Voiculescu exact sequence both hold for smooth crossed products of Fréchet algebras by $\mathbb{R}$ and $\mathbb{Z}$ respectively. We also obtain the same results for L^1-crossed products of Banach algebras by $\mathbb{R}$ and $\mathbb{Z}$.

INTRODUCTION

In [3], Connes proved that the K-theory $K_*(C^*(\mathbb{R}, A))$ is isomorphic to $K_{*+1}(A)$ for any continuous action of $\mathbb{R}$ on a C^*-algebra A. In this paper, we prove the analog of this result for the representable K-theory of smooth crossed products for actions of $\mathbb{R}$ on Fréchet algebras. In order that the representable K-theory of the Fréchet algebra A be defined, we assume that all our Fréchet algebras are locally m-convex. In order to ensure that the crossed products are again locally m-convex, we assume that our actions are m-tempered (Definition 1.1.3). No other restrictions are necessary. As a corollary, we obtain the Pimsner-Voiculescu exact sequence for actions of $\mathbb{Z}$ on Fréchet algebras. As further corollaries, we compute the K-theory of the L^1 crossed products for isometric actions of $\mathbb{R}$ and $\mathbb{Z}$ on arbitrary Banach algebras.

Our smooth crossed products $\mathscr{S}(\mathbb{R}, A)$ and $\mathscr{S}(\mathbb{Z}, A)$ are the standard sets of A-valued Schwartz functions on $\mathbb{R}$ and $\mathbb{Z}$ respectively, with convolution multiplication. (In the language of [12 and 13], these are just the smooth crossed products of Schwartz functions which vanish rapidly with respect to the gauge $\sigma(x) = |x|$ on $\mathbb{Z}$ or $\mathbb{R}$. In a future paper, we hope to consider the K-theory of smooth crossed products defined using stronger rapid vanishing conditions.)

Fréchet algebras occurring as dense subalgebras of C^*-algebras have been important in many recent results in C^*-algebra theory. For example, the theory of differential geometry on a noncommutative space [4] requires the use of "differentiable structures" for these noncommutative spaces, or some sort of algebra of "differentiable functions" on the noncommutative space. Such algebras of functions have usually been provided by a dense Fréchet subalgebra of smooth functions A for which the K-theory $K_*(A)$ is the same as the K-theory of the C^*-algebra $K_*(B)$ (see for example [1, 2, 7, 9], the recent works of Baum and

Connes, Blackadar and Cuntz, R. Ji, P. Jolissaint, V. Nistor and many others). In [9 and 7], methods are given to compute the cyclic cohomology [4] of smooth products of Fréchet algebras by $\mathbb{R}$ and $\mathbb{Z}$.

If the Fréchet algebras are spectral invariant dense subalgebras of C^*-algebras, and the smooth crossed products are spectral invariant in the corresponding C^* crossed products, then our results actually follow from the corresponding C^*-algebra results, since the inclusions of the dense subalgebras into their respective C^*-algebras are all isomorphisms on K-theory. However, in [9] no spectral invariance assumptions are made on the algebra, and in [7] the Fréchet algebra need not be given as a dense subalgebra of a Banach algebra. We also make none of these assumptions. We show that the K-theory of the smooth crossed product is always related to that of the original algebra in the expected way, under essentially the minimum assumptions needed to make sure that all the Schwartz crossed products are m-convex Fréchet algebras, so that a well behaved K-theory can be defined for them.

Our proof of the smooth Connes isomorphism is based on the method of [8]. As a proof of the Connes isomorphism which assumes Bott periodicity, our proof actually reduces in the C^*-algebra case to an even simpler proof than the one in [8], because the continuous fields have been replaced by ordinary short exact sequences. For more details on the relation with [8], see the beginning of §1.

In §1.1, we define a map from the representable K-theory $RK_*(A)$ of A to $RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha))$, where α is a differentiable action of $\mathbb{R}$ on A. In §1.2, we show that this map is an isomorphism. In §1.3, we show that the hypothesis that α be a differentiable action can be dropped, as long as α is assumed strongly continuous. We also prove the results for the L^1 crossed product, using spectral invariance arguments.

In §2, we define the smooth mapping torus and use this to show that the Pimsner-Voiculescu exact sequence holds for smooth crossed products by $\mathbb{Z}$, following the method of Theorem 10.2.1 of [1]. In §3, we give some applications and examples, including some nonstandard ones.

We will use the notation $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{T}$ for the reals, complexes, natural numbers (including zero), integers, and the circle group respectively. The symbols $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}(\mathbb{Z})$ will denote the standard Fréchet space of Schwartz functions on $\mathbb{R}$ and $\mathbb{Z}$ respectively. All Fréchet spaces will be assumed locally convex, and to simplify terminology we will call a locally m-convex Fréchet algebra a Fréchet algebra. (m-convex means that the topology can be given by a family of submultiplicative seminorms.) The tensor product $\otimes$ will always mean the completed projective tensor product of two Fréchet spaces. By an action α of a group G on a Fréchet algebra A we will mean a group homomorphism $\alpha: G \to \text{Aut}(A)$. The notation A^+ will mean "A with unit adjoined", even if the algebra A already has a unit.

When we say "K-theory" or "isomorphism on K-theory," we will mean with regard to the representable K-functor (denoted by RK) of [10].

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1. SMOOTH CROSSED PRODUCTS BY $\mathbb{R}$

In this section we prove that the Thom isomorphism holds for smooth crossed products $\mathscr{S}(\mathbb{R}, A, \alpha)$, where α is any strongly continuous (not necessarily smooth), m-tempered action of $\mathbb{R}$ on a Fréchet algebra A.

Before we begin §1.1, we give a brief discussion of the development of our method of proof. In [8], a map from the suspension of A to the C^*-crossed product is defined roughly as follows. For $s \in [0, 1]$, let $C^*(\mathbb{R}, A, \alpha_s)$ be the C^*-crossed product of A with $\mathbb{R}$, with action $t \mapsto \alpha_{ts}$. For $s = 0$, this is just the suspension of A. These algebras fit together to form a continuous field of C^*-algebras over $[0, 1]$. Let u be some invertible in the unitization of the suspension of A. Find a continuous section $s \mapsto u_s$ through u. Then for s sufficiently small, u_s will be invertible in $C^*(\mathbb{R}, A, \alpha_s)^+$. The argument for this depends on the fact that a Banach algebra has an open invertible group. Then one notes that for $s > 0$, the fibers $C^*(\mathbb{R}, A, \alpha_s)$ are all isomorphic as algebras, so that u_s in fact defines an invertible in $C^*(\mathbb{R}, A, \alpha)^+$. This then defines a map from $K_1(SA)$ to $K_1(C^*(\mathbb{R}, A, \alpha))$. This map is composed with the Bott map to get the Thom map from $K_0(A)$ to $K_1(C^*(\mathbb{R}, A, \alpha))$.

The problem with imitating this for smooth crossed products is that $\mathscr{S}(\mathbb{R}, A, \alpha_s)^+$ may *not* have an open group of invertibles. So we would not know that for small s the element u above is invertible in $\mathscr{S}(\mathbb{R}, A, \alpha_s)^+$. Moreover, the decomposition of $\mathscr{S}(\mathbb{R}, A, \alpha_s)$ as an inverse limit of Banach algebras may vary as s varies, making it awkward to work on the Banach algebra level. We use the following device to overcome this. We let $\mathbb{R}$ act on the pointwise multiplication algebra $C^\infty([0, 1], A)$ by $\beta_t(f)(s) = \alpha_{st}(f(s))$. Then $\mathscr{S}(\mathbb{R}, C^\infty([0, 1], A), \beta)$ is a Fréchet algebra which serves as a substitute for the continuous field $\{\mathscr{S}(\mathbb{R}, A, \alpha_{ts})\}_{s \in [0, 1]}$. There are natural maps from $\mathscr{S}(\mathbb{R}, C^\infty([0, 1], A), \beta)$ to the suspension $\mathscr{S}(\mathbb{R}) \otimes A$ and to $\mathscr{S}(\mathbb{R}, A, \alpha)$ given by evaluation at 0 and 1 respectively. We show that the former map can be inverted in Corollary 1.1.12 below. Thus we define our Thom map for smooth crossed products to be this inverse composed with the second map of evaluation at 1.

Our proof in §1.2 that this composition is an isomorphism proceeds much like the proof of [8], using commutative diagrams and showing that one path of the diagram is essentially the identity map, so that another path of the diagram (which has the Thom map as one of the edges) must be also.

We conclude this discussion by looking at the possibility of imitating other proofs of the Thom isomorphism for C^*-crossed products by $\mathbb{R}$, in particular [3 and 1, §10.9]. One essential part of the proof in [3] is as follows. One has the crossed product $C^*(\mathbb{R}, A, \alpha)$, and an idempotent e in A. One then wants to define an exterior equivalent action γ of $\mathbb{R}$ on A, such that the crossed products $C^*(\mathbb{R}, A, \alpha)$ and $C^*(\mathbb{R}, A, \gamma)$ are isomorphic, and γ leaves e fixed. This exterior equivalence is implemented by a multiplier on $C^*(\mathbb{R}, A, \alpha)$ given by the formula

$$(*) \qquad u_t = \sum_{n=0}^{\infty} i^n \int_{0 \le s_1 \le \cdots \le s_n \le t} \alpha_{s_1}(x) \cdots \alpha_{s_n}(x) \, ds_1 \cdots ds_n ,$$

where x is some selfadjoint element of A depending on e. This formula for u_t converges in the norm on A, essentially because the series $\sum_{n=0}^{\infty} t^n/n!$

converges for any t. Also, u_t is a unitary element of the C^*-algebra A for each t. (See proof of Proposition 4 of [3, §II].) However, if we try to imitate this in our setting, when A is a Banach or Fréchet algebra, and we use the smooth crossed product in place of the C^*-crossed product, then it is not clear that $(*)$ gives us a nice enough u_t. In particular we need $\|u_t\|$ to be bounded by some polynomial in t, whereas the formula $(*)$ seems to give an exponential bound at best.

Imitating Blackadar's proof [1, §10.9] seems to require an explicit formula for a smooth version of the function F in [1, Lemma 10.9.3]. The spectral invariance of the smooth compact operators $\mathscr{K}^\infty$ in the compact operators $\mathscr{K}$ should at least come very close to ensuring the existence of such an F, but an explicit formula seems to be difficult to find.

1.1 Definitions and preliminaries.

The main purpose of this section is to construct a map ε_0 of Fréchet algebras associated with a smooth action of $\mathbb{R}$, and to prove that it is an isomorphism on K-theory. This is necessary because $(\varepsilon_0)_*^{-1}$ is used in the next section in our definition of the Thom map from the K-theory of A to the K-theory of the smooth crossed product.

1.1.1 Definition.

Let A be a Fréchet algebra. Then the *smooth interval algebra* $I_\infty A$ of A is the Fréchet algebra $C^\infty([0, 1], A)$ of all C^∞ functions f from $[0, 1]$ to A with the topology of uniform convergence of each derivative of f in each continuous seminorm on A. The *smooth cone* of A is

$$C_\infty A = \{f \in I_\infty A \mid f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N}\},$$

where $f^{(n)}$ denotes the nth derivative of f. We give $C_\infty A$ the topology inherited from $I_\infty A$.

1.1.2 Lemma.

(1) *For each k, $n \in \mathbb{N}$, the seminorm*

$$\|f\|'_{k,n,m} = \sup_{r \in [0,1]} \|r^{-k} f^{(n)}(r)\|_m,$$

where $\|\ \|_m$ is any continuous seminorm on A, is well defined and continuous on $C_\infty A$.

(2) *The Fréchet spaces $C_\infty \mathbb{C}$ and $I_\infty \mathbb{C}$ are nuclear topological vector spaces.*

(3) *We have isomorphisms of Fréchet spaces $C_\infty A \cong C_\infty \mathbb{C} \otimes A$ and $I_\infty A \cong I_\infty \mathbb{C} \otimes A$.*

(4) *The Fréchet algebra $C_\infty A$ is a closed two-sided ideal in $I_\infty A$ and $I_\infty A / C_\infty A \cong A[[z]]$, the Fréchet algebra of formal power series in z with coefficients in A, with the topology of pointwise convergence of the coefficients. If $z(r) = r$ is the identity function in $I_\infty \mathbb{C}$, then this isomorphism sends $z^m I_\infty A / C_\infty A$ onto $z^m A[[z]]$ for each $m \in \mathbb{N}$.*

Proof. Using the integral form of the remainder for the Taylor polynomial, for any C^∞ function $g : [0, 1] \to A$ and $r \in [0, 1]$ we obtain

$$\left\| g(r) - g(0) - \cdots - \frac{g^{(k-1)}(0)}{(k-1)!} r^{k-1} \right\|_m$$

$$= \left\| \int_0^r \frac{g^{(k)}(s)}{(k-1)!} (r-s)^{k-1} \, ds \right\|_m \leq \frac{r^k}{k!} \sup_{s \in [0,r]} \|g^{(k)}(s)\|_m.$$

Let $f \in C_\infty A$, and put $g = f^{(n)}$. Since the derivatives of $f^{(n)}$ all vanish at zero, we obtain

$$\|r^{-k} f^{(n)}(r)\|_m \le \frac{1}{k!} \sup_{s \in [0,1]} \|f^{(n+k)}(s)\|_m.$$

Taking the sup over $r \in [0, 1]$ on the left yields the continuity of $\| \ \|'_{k,n,m}$.

To prove (2), note that $I_\infty \mathbb{C}$ is nuclear. (For example, combine [14, Theorem 51.5] and [2, Proposition 5.2.1].) Then the space $C_\infty \mathbb{C}$ is a closed subspace of the nuclear locally convex space $I_\infty \mathbb{C}$ and hence is nuclear [14, Proposition 50.1].

For (3), note that the topologies on $I_\infty A$ and $C_\infty A$ are already given by tensor product seminorms, and the tensor products are unique by (2).

We prove (4). Define $\pi \colon I_\infty \mathbb{C} \to \mathbb{C}[[z]]$ by

$$\pi(f) = \left(f(0), f^{(1)}(0), \frac{1}{2!} f^{(2)}(0), \dots \right).$$

Standard differentiation rules show that π is a homomorphism, and it is trivially continuous. It is known (see [14, Theorem 38.1]) that for every sequence $(c_0, c_1, c_2, \dots)$ of complex numbers, there exists a C^∞ complex valued function f on $[0, 1]$ such that $f^{(n)}(0) = c_n$ for all $n \in \mathbb{N}$. This proves the surjectivity of π.

Clearly $\mathrm{Ker}(\pi) = C_\infty \mathbb{C}$. Therefore π defines a continuous bijection $\bar{\pi} : I_\infty \mathbb{C} / C_\infty \mathbb{C} \to \mathbb{C}[[z]]$. Since both these algebras are Fréchet, the open mapping theorem for Fréchet spaces [14, Theorem 17.1] implies that $\bar{\pi}$ is a homeomorphism. (Note that in [14], a "homomorphism" is actually a quotient map; see the beginning of Chapter 17.)

In the notation of the lemma, $\pi(z^m f) = z^m \pi(f)$, and from this one easily sees that $\bar{\pi}$ maps $z^m I_\infty \mathbb{C} / C_\infty \mathbb{C}$ onto $z^m \mathbb{C}[[z]]$.

To prove (4), it suffices to show that we can replace $\mathbb{C}$ with A. The map $I_\infty A / C_\infty A \to A[[z]]$ is just the projective tensor product of $\bar{\pi}$ with the identity map on A. (Note that $\mathbb{C}[[z]]$ is nuclear because it is a product of nuclear Fréchet spaces [14, Proposition 50.1]; thus we have $A[[z]] \cong \mathbb{C}[[z]] \otimes A$.) Since the projective tensor product of quotient maps of Fréchet spaces is a quotient map [14, Proposition 43.9], we have the result. $\square$

1.1.3 Definition [12, §3]. We say that an action α of $\mathbb{R}$ on a Fréchet algebra A is *m-tempered* if there exists a family of submultiplicative seminorms $\{\| \ \|_m\}_{m=0}^\infty$ giving the topology of A, such that for every $m \in \mathbb{N}$, there is a polynomial poly such that

$$\|\alpha_r(a)\|_m \le \mathrm{poly}(r) \|a\|_m,$$

for $a \in A$ and $r \in \mathbb{R}$. (This notion of an *m*-tempered action is a special case of [12, Definition 3.1.1] with the gauge $\sigma(r) = |r|$ on $\mathbb{R}$.) For use in §2, note that the same definition make sense for $\mathbb{Z}$ or $\mathbb{T}$ in place of $\mathbb{R}$. In the case of $\mathbb{T}$, the polynomial may be replaced by a constant. If α is an *m*-tempered action of $\mathbb{R}$ on A, then the Fréchet space $\mathscr{S}(\mathbb{R}) \otimes A$ is an (*m*-convex) Fréchet algebra under the convolution product

$$(f * g)(r) = \int_{\mathbb{R}} f(s) \alpha_s(g(r-s)) \, ds,$$

where f, $g \in \mathscr{S}(\mathbb{R}) \otimes A$ and $r \in \mathbb{R}$ [12, Theorem 3.1.7]. We denote this Fréchet algebra by $\mathscr{S}(\mathbb{R}, A, \alpha)$, and call it the smooth crossed product.

1.1.4 Lemma. *If*

$$0 \to J \to A \to B \to 0$$

is a short equivariant exact sequence of Fréchet algebras with m-tempered actions γ, α, and β of $\mathbb{R}$, then

$$0 \to \mathscr{S}(\mathbb{R}, J, \gamma) \to \mathscr{S}(\mathbb{R}, A, \alpha) \to \mathscr{S}(\mathbb{R}, B, \beta) \to 0$$

is an exact sequence of Fréchet algebras.

Proof. We can disregard the algebra structure; this then reduces to the exactness of tensoring with the nuclear Fréchet space $\mathscr{S}(\mathbb{R})$ [10, Theorem 2.3(3b)]. □

1.1.5 Definition. Let α be an action of $\mathbb{R}$ on A. If for every $a \in A$ the map $r \mapsto \alpha_r(a)$ is C^∞, we will say that α is a *smooth action*. If α is smooth and m-tempered, we define the *interval action* $I_\infty(\alpha)$ of $\mathbb{R}$ on $I_\infty A$ by

$$I_\infty(\alpha)_s(f)(r) = \alpha_{rs}(f(r))$$

for $s \in \mathbb{R}$, $f \in I_\infty A$, and $r \in [0, 1]$. The *cone action* $C_\infty(\alpha)$ is defined to be the restriction of $I_\infty(\alpha)$ to $C_\infty A$.

1.1.6 Lemma. *Let A, α, $I_\infty(\alpha)$, $C_\infty(\alpha)$ be as in the previous definition. Then $I_\infty(\alpha)$ and $C_\infty(\alpha)$ are well defined, smooth, and m-tempered actions of $\mathbb{R}$ on $I_\infty A$ and $C_\infty A$ respectively. Furthermore, if z is the identity function $z(r) = r$, then for each $n \in \mathbb{N}$, the subspace $z^m I_\infty A$ is a closed $I_\infty(\alpha)$-invariant ideal in $I_\infty A$.*

Proof. Let $\| \ \|_m$ be submultiplicative seminorms giving the topology of A. Let D denote the differential operator $Df(r) = f^{(1)}(r)$ for $f \in I_\infty A$, and let $\widetilde{D}$ denote the differential operator

$$\widetilde{D}f(r) = \lim_{s \to 0} \frac{\alpha_s(f(r)) - f(r)}{s}.$$

Note that D and $\widetilde{D}$ commute. The topology on $I_\infty A$ can then be given by the seminorms

$$\|f\|'_m = \max_{p+q \leq m} \|\widetilde{D}^p D^q f\|_m = \max_{p+q \leq m} \left(\sup_{r \in [0, 1]} \|\widetilde{D}^p D^q f(r)\|_m \right).$$

The seminorms $\| \ \|'_m$ are easily seen to be submultiplicative modulo constants. Note that

$$\|\widetilde{D}^p D^q I_\infty(\alpha)_s(f)\|_m = \sup_{r \in [0, 1]} \|\widetilde{D}^p D^q \alpha_{sr}(f(r))\|_m,$$

where D acts on r. Let $p + q \leq m$. By the chain rule and since $\widetilde{D}$ commutes with α, this is bounded by

$$C \sum_{u+t \leq q} s^u \|\alpha_{sr}(\widetilde{D}^{p+u} D^t f(r))\|_m \leq \text{poly}(s) \|f\|'_m$$

where we used the m-temperedness of the action of $\mathbb{R}$ on A. It follows that $\|I_\infty(\alpha)_s(f)\|'_m \leq \text{poly}(s) \|f\|'_m$, so $I_\infty(\alpha)$ is m-tempered. The smoothness of $I_\infty(\alpha)$ is straightforward using the chain rule.

To prove the statement about $C_\infty A$, it suffices to show that $I_\infty A$ leaves $C_\infty A$ invariant. This will follow from the last statement, since $C_\infty A = \bigcap_{m=0}^\infty z^m I_\infty A$.

We therefore prove the last statement. Invariance follows from $\widetilde{I}_\infty(\alpha)(z^m f) = z^m I_\infty(\alpha)(f)$. That $z^m I_\infty A$ is an ideal in $I_\infty A$ is obvious. Closedness follows from Lemma 1.1.2(4). $\square$

1.1.7 Lemma. *Let α be a smooth m-tempered action of $\mathbb{R}$ on a Fréchet algebra A. Then*

$$\mathscr{S}(\mathbb{R}, C_\infty A, C_\infty(\alpha)) \cong C_\infty(\mathscr{S}(\mathbb{R}, A, \alpha))$$

as Fréchet algebras.

Proof. Recall that as a Fréchet space, $B_1 = \mathscr{S}(\mathbb{R}, C_\infty A, C_\infty(\alpha))$ is isomorphic to $\mathscr{S}(\mathbb{R}) \otimes C_\infty \mathbb{C} \otimes A$ with multiplication

$$(*) \qquad \begin{aligned} (f *_1 g)(s, r) &= \left(\int_\mathbb{R} f(t) C_\infty(\alpha)_t(g(s - t)) \, dt \right)(r) \\ &= \int_\mathbb{R} f(t, r) \alpha_{rt}(g(s - t, r)) \, dt \end{aligned}$$

for $s \in \mathbb{R}$ and $r \in [0, 1]$. By commuting $\mathscr{S}(\mathbb{R})$ and $C_\infty \mathbb{C}$ in the tensor product, we may also identify $B_2 = C_\infty(\mathscr{S}(\mathbb{R}, A, \alpha))$ with $\mathscr{S}(\mathbb{R}) \otimes C_\infty \mathbb{C} \otimes A$, but with multiplication

$$(**) \qquad (f *_2 g)(s, r) = \int_\mathbb{R} f(t, r) \alpha_t(g(s - t, r)) \, dt$$

for $s \in \mathbb{R}$ and $r \in [0, 1]$. Define $\gamma : B_1 \to B_2$ by $(\gamma f)(s, r) = r^{-1} f(r^{-1} s, r)$. We must show this is well defined and continuous. By Lemma 1.1.2(1), we may topologize $\mathscr{S}(\mathbb{R}) \otimes C_\infty \mathbb{C} \otimes A$ by the seminorms

$$\|f\|_{p, q, l, n, m} = \sup_{s \in \mathbb{R}, r \in [0, 1]} \omega(s)^p r^{-q} \|D^l \widetilde{D}^n f(s, r)\|_m,$$

where $\omega(s) = 1 + |s|$, and D and $\widetilde{D}$ are derivatives in s and r respectively. (Note that they commute.)

By the chain rule and the product rule,

$$\begin{aligned} \|D^l \widetilde{D}^n (\gamma f)(s, r)\|_m &= \|D^l \widetilde{D}^n r^{-1} f(r^{-1} s, r)\|_m \\ &= \|\widetilde{D}^n r^{-l-1} (D^l f)(r^{-1} s, r)\|_m \leq C \sum_{i, j, k \leq N} r^{-i} \|(D^j \widetilde{D}^k f)(r^{-1} s, r)\|_m, \end{aligned}$$

for some large N not depending on f. Now if we multiply a generic summand by $\omega(s)^p r^{-q}$ and take the sup over $r, s \in \mathbb{R}$, we get something bounded by

$$\begin{aligned} \sup_{s, r} \omega(s)^p r^{-q-i} \|(D^j \widetilde{D}^k f)(r^{-1} s, r)\|_m &= \sup_{s, r} \omega(sr)^p r^{-q-i} \|(D^j \widetilde{D}^k f)(s, r)\|_m \\ &\leq \sup_{s, r} \omega(s)^p r^{-q-i} \|(D^j \widetilde{D}^k f)(s, r)\|_m = \|f\|_{p, q+i, j, k, m}, \end{aligned}$$

where we replaced s by sr and used $\omega(sr) \leq \omega(s)$ for $r \in [0, 1]$. From these estimates, we see that γ is a well-defined map and that

$$\|\gamma f\|_{p, q, k, n, m} \leq C \sum_{i, j, k \leq N} \|f\|_{p, q+i, j, k, m},$$

so γ is continuous. We easily check that $\gamma(f *_1 g) = \gamma f *_2 \gamma g$ using (*) and (**), so it remains to show that γ is a bijection, and that γ^{-1} continuous.

Define $\tau \colon B_2 \to B_1$ by $(\tau f)(s, r) = rf(rs, r)$. Clearly τ is an inverse for γ as only as τf is in B_1 for every $f \in B_2$. We have

$$\|D^l \widetilde{D}^n (\tau f)(s, r)\|_m \le C \sum_{j \le l, k \le n} \text{poly}(r) \|(D^j \widetilde{D}^k f)(sr, r)\|_m.$$

Using $f \in B_2$ and $\omega(r^{-1}s) \le r^{-1}\omega(s)$, one easily checks that $\|\tau f\|_{p,q,l,n,m} < \infty$ for all p, q, l, n, $m \in \mathbb{N}$. Hence $\tau f \in B_1$ and τ is an inverse for γ. By the open mapping theorem for Fréchet spaces, γ is an isomorphism and we have proved Lemma 1.1.7. $\square$

1.1.8 *Remark.* Note that the proof above does not work if one uses $\{f \in I_\infty A \mid f(0) = 0\}$ in place of $C_\infty A$.

Before we prove our first result on K-theory, we prove a lemma on spectral invariance. This will be used repeatedly throughout the paper to show that various dense inclusions induce isomorphisms on K-theory. We say that a subalgebra A of an algebra B is *spectral invariant* in B if the invertible elements of the unitization A^+ are precisely the invertible elements of B^+ which lie in A^+.

1.1.9 Spectral invariance lemma.

(1) *Let A be a dense Fréchet subalgebra of a Banach algebra B. If A is spectral invariant in B, then the inclusion map $A \hookrightarrow B$ induces an isomorphism $RK_*(A) \cong RK_*(B)$.*

(2) *Let A be a dense Fréchet subalgebra of a Fréchet algebra B. Assume that we can write A and B as inverse limits of Banach algebras A_n and B_n respectively, where A_n is dense in B_n for all n, and the inclusions $A \subseteq A_n$, $B \subseteq B_n$ are all dense. If each A_n is spectral invariant in B_n, then the inclusion $A \hookrightarrow B$ induces an isomorphism $RK_*(A) \cong RK_*(B)$.*

Proof. For the first item, by Theorem A.2.1 in the appendix of [2], we know that the inclusion induces an isomorphism $K_*(A) \cong K_*(B)$. Since A^+ and B^+ have open groups of invertibles, $RK_*(A) \cong K_*(A)$ and $RK_*(B) \cong K_*(B)$ [10, Theorem 7.7]. Thus $RK_*(A) \cong RK_*(B)$.

For the second item, again by [2, Theorem A.2.1] we have $K_*(A_n) \cong K_*(B_n)$ for all $n \in \mathbb{N}$. Since A_n and B_n are Banach algebras, their K_* is the same as their RK_* [10, Corollary 7.8]. Applying the Milnor $\varprojlim^1$-sequence [10, Theorem 6.5], one concludes that $RK_*(A) \cong RK_*(B)$. $\square$

We let $\mathscr{S}(\mathbb{R}) \otimes A$ denote the Fréchet algebra of Schwartz functions from $\mathbb{R}$ to A with convolution multiplication, for the trivial action of $\mathbb{R}$ on A. (Note that it is isomorphic to the same set of functions with pointwise multiplication via the Fourier transform.) We find $\mathscr{S}(\mathbb{R}) \otimes A$ a more convenient suspension than, say, $C_0(\mathbb{R}, A)$. Part (3) of the next lemma shows that it behaves the same way for K-theory.

1.1.10 **Lemma.** *Let A be a Fréchet algebra. Then*
   (1) *Evaluation at $r \in [0, 1]$ induces an isomorphism*

$$(\text{ev}_r)_* \colon RK_*(I_\infty A) \to RK_*(A),$$

which does not depend on r.

(2) $RK_*(C_\infty A) = 0$.

(3) *There is a natural isomorphism* $RK_*(A) \cong RK_{*+1}(\mathscr{S}(\mathbb{R}) \otimes A)$.

Proof. (1) The maps ev_r are all homotopic, so the maps $(\mathrm{ev}_r)_*$ are all equal. To see that they are isomorphisms, it suffices to show that one of them, say ev_0, is a homotopy equivalence. The homotopy inverse $\varphi\colon A \to I_\infty A$ is given by $\varphi(a)(r) = a$. The homotopy from $\varphi \circ \mathrm{ev}_0$ to $\mathrm{id}_{I_\infty A}$ is given by $\psi_t(f)(r) = f(tr)$ for t, $r \in [0, 1]$. One easily verifies that everything is in fact smooth and continuous in the C^∞ topology.

(2) The identity map of $C_\infty A$ is homotopic to the zero map via $\psi_t(f)(r) = f(tr)$.

(3) Let $\{\| \ \|_m\}_{m=0}^\infty$ be an increasing sequence of submultiplicative seminorms which define the topology of A, and let A_m denote the Banach algebra obtained by completing $A/\mathrm{Ker}(\| \ \|_m)$. Set

$$B_m = \left\{ f \in C^m(\mathbb{R}, A_m) \middle| \|f\|_m' = \max_{0 \le k \le m} \left(\sup_{s \in \mathbb{R}} (1 + |s|)^m |f^{(k)}(s)| \right) < \infty \right\}.$$

Then B_m with pointwise multiplication is a Banach algebra in a norm equivalent to $\| \ \|_m'$, and B_m is a dense subalgebra of the Banach algebra $C_m = C_0(\mathbb{R}, A_m)$ of continuous functions from $\mathbb{R}$ to A_m which vanish at infinity. It is easily checked that B_m is spectrally invariant in C_m [13, (2.3)–(2.4)]. Also, $\varprojlim C_m \cong C_0(\mathbb{R}, A)$, which is the usual suspension of A, and $\varprojlim B_m \cong \mathscr{S}(\mathbb{R}) \otimes A$, where $\mathscr{S}(\mathbb{R})$ is taken with pointwise multiplication. The Fourier transform shows that this algebra is isomorphic to the one obtained using convolution on $\mathscr{S}(\mathbb{R})$. Therefore, using Lemma 1.1.9(2) in the first step and Bott periodicity [10, Theorem 5.5] in the second step, we obtain

$$RK_{*+1}(\mathscr{S}(\mathbb{R}) \otimes A) \cong RK_{*+1}(C_0(\mathbb{R}, A)) \cong RK_*(A). \quad \square$$

Item (3) in the previous lemma can also be proved using homotopy invariance results of A. M. Davie [6], instead of using spectral invariance.

We let $\mathscr{K}^\infty$ denote the Fréchet algebra of rapidly vanishing complex valued matrices on $\mathbb{Z} \times \mathbb{Z}$. This is the smooth version of the compact operators used in [10, §2 and 13, §5].

1.1.11 Lemma. *Let* α *be a smooth m-tempered action of* $\mathbb{R}$ *on the Fréchet algebra* A. *Let* β *be the action on the quotient* $I_\infty A/C_\infty A \cong A[[z]]$ *determined by* $I_\infty(\alpha)$. *Then* $\mathrm{ev}_0\colon I_\infty A \to A$ *determines an isomorphism*

$$RK_*(\mathscr{S}(\mathbb{R}, I_\infty A/C_\infty A, \beta)) \cong RK_*(\mathscr{S}(\mathbb{R}) \otimes A).$$

Proof. Note that ev_0 is equivariant for the action $I_\infty(\alpha)$ on $I_\infty A$ and the trivial action τ on A. Therefore we obtain

$$\overline{\mathrm{ev}}_0\colon I_\infty A/C_\infty A \to A$$

and

$$\overline{\varepsilon}_0\colon \mathscr{S}(\mathbb{R}, I_\infty A/C_\infty A, \beta) \to \mathscr{S}(\mathbb{R}, A, \tau) \cong \mathscr{S}(\mathbb{R}) \otimes A.$$

We have to show that $(\overline{\varepsilon}_0)_*$ is an isomorphism. By Lemmas 1.1.6 and 1.1.2(4), we have $I_\infty A/C_\infty A \cong A[[z]]$, and the induced action γ of $\mathbb{R}$ on $A[[z]]$ satisfies $\gamma_s(z^m A[[z]]) \subseteq z^m A[[z]]$ for all $m \in \mathbb{N}$. It is easily verified that if $a = a_0 +$

$a_1 z + \cdots \in A[[z]]$, then $\gamma(a) = b_0 + b_1 z + \cdots$ satisfies $b_0 = a_0$. Thus $a \mapsto a_0$ is an equivariant map from $(A[[z]], \gamma)$ to A with the trivial action.

Let $B = \mathscr{S}(\mathbb{R}, A[[z]], \gamma)$, and let B_m be the closed ideal $\mathscr{S}(\mathbb{R}, z^m A[[z]], \gamma)$. Then $B = B_0$, and $B/B_1 \cong SA$. We have to prove that the quotient map $B \to B/B_1$ is an isomorphism on K-theory. Considering the long exact sequence in K-theory [10, Theorem 6.1], it suffices to show that $RK_*(B_1) = 0$.

We first prove $RK_1(B_1) = 0$. This is equivalent to showing that the invertible group $\mathrm{inv}((\mathscr{K}^\infty \otimes B_1)^+)$ is connected. As a topological vector space (TVS), we have $zA[[z]] \cong \prod_{m=1}^\infty z^m A[[z]]/z^{m+1}A[[z]]$. Further, as TVS's we have $\mathscr{S}(\mathbb{R}, D, \delta) \cong \mathscr{S}(\mathbb{R}) \otimes D$ for any D and δ. Hence $B_1 \cong \prod_{m=1}^\infty B_m/B_{m+1}$ as TVS's. It follows that $\mathscr{K}^\infty \otimes B_1 \cong \prod_{m=1}^\infty (\mathscr{K}^\infty \otimes B_m)/(\mathscr{K}^\infty \otimes B_{m+1})$ as TVS's.

For $m, n \in \mathbb{N}$, we have $(z^m A[[z]])(z^n A[[z]]) \subseteq z^{m+n} A[[z]]$. The fact that $z^{m+n} A[[z]]$ is closed now implies that $B_m B_n \subseteq B_{m+n}$, so
$$(\mathscr{K}^\infty \otimes B_m)(\mathscr{K}^\infty \otimes B_n) \subseteq \mathscr{K}^\infty \otimes B_{m+n}.$$

We will now show that if $\eta \in \mathbb{C} - \{0\}$ and $x \in \mathscr{K}^\infty \otimes B_1$ then $\eta - x$ is invertible. This will clearly imply that $\mathrm{inv}((\mathscr{K}^\infty \otimes B_1)^+)$ is connected. Without loss of generality, let $\eta = 1$. Note that $x^n \to 0$ in $\mathscr{K}^\infty \otimes B_1$, because the image of x^n in $\prod_{m=1}^\infty (\mathscr{K}^\infty \otimes B_m)/(\mathscr{K}^\infty \otimes B_{m+1})$ is zero in the first $n-1$ factors. Furthermore, the sum $\sum_{n=1}^\infty x^n$ converges in $\mathscr{K}^\infty \otimes B_1$, since $\sum_{n=N}^{N+p} x^n$ is zero in the first $N-1$ factors of $\prod_{m=1}^\infty (\mathscr{K}^\infty \otimes B_m)/(\mathscr{K}^\infty \otimes B_{m+1})$, and the topology is given by pointwise convergence in each factor. Therefore $1 - x$ is invertible with inverse $1 + \sum_{n=1}^\infty x^n$.

It remains to show that $RK_0(B_1) = 0$. If we replace A by $C^\infty(\mathbb{T}) \otimes B$, it is easy to see (using the nuclearity of $C^\infty(\mathbb{T})$) that this replaces B_1 by $C^\infty(\mathbb{T}) \otimes B_1$. Therefore we know from the above that $RK_1(C^\infty(\mathbb{T}) \otimes B_1) = 0$. By the estimates [13, (2.3)–(2.4)], the inclusion $C^\infty(\mathbb{T}) \otimes B_1 \hookrightarrow C(\mathbb{T}, B_1)$ is an isomorphism on K-theory. Let $SB_1 = \{f \in C(\mathbb{T}, B_1) | f(1) = 0\}$. Then from the split exact sequence
$$0 \to SB_1 \to C(\mathbb{T}, B_1) \to B_1 \to 0,$$
and Bott periodicity [10, Theorem 5.5], we have $RK_1(C(\mathbb{T}, B_1)) = RK_0(B_1) \oplus RK_1(B_1)$. Hence $RK_0(B_1) = 0$.   □

1.1.12 Corollary. *The map*
$$\varepsilon_0 \colon \mathscr{S}(\mathbb{R}, I_\infty A, I_\infty(\alpha)) \to \mathscr{S}(\mathbb{R}) \otimes A,$$
induced on the crossed products by $\mathrm{ev}_0 \colon I_\infty A \to A$, *is an isomorphism on K-theory.*

Proof. Using the notation of the previous lemma, we can factor ε_0 as
$$\mathscr{S}(\mathbb{R}, I_\infty A, I_\infty(\alpha)) \to \mathscr{S}(\mathbb{R}, I_\infty A/C_\infty A, \beta) \to \mathscr{S}(\mathbb{R}) \otimes A.$$

The kernel of the first map is $\mathscr{S}(\mathbb{R}, C_\infty A, C_\infty(\alpha))$ (by Lemma 1.1.4), so Lemmas 1.1.7 and 1.1.10(2) imply that its K-theory is 0. The long exact sequence for K-theory [10, Theorem 6.1] therefore shows that the first map is an isomorphism on K-theory. The second map is an isomorphism on K-theory by Lemma 1.1.11.   □

1.2. The Thom map θ is an isomorphism. We let β (or, when necessary, β_A) denote the Bott periodicity isomorphism $RK_*(A) \to RK_{*+1}(\mathscr{S}(\mathbb{R}) \otimes A)$, as obtained in Lemma 1.1.10(3).

Throughout this section, we shall use $I(\alpha)$ to denote $I_\infty(\alpha)$.

1.2.1 Definition. Let α be a smooth m-tempered action of $\mathbb{R}$ on a Fréchet algebra A. We define $F(A, \alpha) = \mathscr{S}(\mathbb{R}, I_\infty A, I(\alpha))$. (This corresponds to the algebra of sections of the continuous field of [8].) We let

$$\varepsilon_0 \colon F(A, \alpha) \to \mathscr{S}(\mathbb{R}) \otimes A$$

be as in Corollary 1.1.12, and we let

$$\varepsilon_1 \colon F(A, \alpha) \to \mathscr{S}(\mathbb{R}, A, \alpha)$$

be the map on the crossed products determined by $\mathrm{ev}_1 \colon I_\infty A \to A$. We then define

$$\theta^{(0)} \colon RK_*(\mathscr{S}(\mathbb{R}) \otimes A) \to RK_*(\mathscr{S}(\mathbb{R}, A, \alpha))$$

by $\theta^{(0)} = (\varepsilon_1)_* \circ (\varepsilon_0)_*^{-1}$. (We sometimes write $\theta_A^{(0)}$ or $\theta_{A,\alpha}^{(0)}$.) We further define the Thom map

$$\theta \colon RK_*(A) \to RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha))$$

by $\theta = \theta^{(0)} \circ \beta$ (or $\theta_A = \theta_A^{(0)} \circ \beta_A$).

1.2.2 Lemma. *The maps $\theta^{(0)}$ and θ are well-defined natural group homomorphisms. Moreover, they both commute with Bott periodicity. That is, the diagram*

$$
\begin{array}{ccc}
RK_*(\mathscr{S}(\mathbb{R}) \otimes A) & \xrightarrow{\ \theta_A^{(0)}\ } & RK_*(\mathscr{S}(\mathbb{R}, A, \alpha)) \\[2mm]
\beta \downarrow & & \downarrow \beta \\[2mm]
RK_{*+1}(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes A) & \xrightarrow{\theta_{\mathscr{S}(\mathbb{R}) \otimes A}^{(0)}} & RK_{*+1}(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}) \otimes A, \mathrm{id} \otimes \alpha))
\end{array}
$$

commutes, and similarly for θ in place of $\theta^{(0)}$.

Proof. Since θ is essentially $\theta^{(0)} \circ \beta$, we need only prove this for $\theta^{(0)}$. Since $\theta^{(0)} = (\varepsilon_1)_* \circ (\varepsilon_0)_*^{-1}$, this follows from the naturality of Bott periodicity and the isomorphism $F(\mathscr{S}(\mathbb{R}) \otimes A, \mathrm{id} \otimes \alpha) \cong \mathscr{S}(\mathbb{R}) \otimes F(A, \alpha)$. $\square$

1.2.3 Lemma. *Let α be an m-tempered action of $\mathbb{R}$ on a Fréchet algebra A. Then the dual action $\hat{\alpha}$ of $\mathbb{R}$ on $\mathscr{S}(\mathbb{R}, A, \alpha)$ is smooth and m-tempered.*

Proof. Recall that $\hat{\alpha}_s(f)(r) = e^{2\pi i r s} f(r)$, where s, $r \in \mathbb{R}$ and $f \in \mathscr{S}(\mathbb{R}, A, \alpha)$. Since differentiating in s only brings down powers of $2\pi i r$ and $\mathscr{S}(\mathbb{R}, A, \alpha)$ consists of Schwartz functions, we see that $\hat{\alpha}$ is a smooth action.

Let $\{\|\ \|_m\}$ be a family of submultiplicative seminorms giving the topology on the convolution algebra of L^1-rapidly vanishing functions $L_1^{|\ |}(\mathbb{R}, A, \alpha)$ from $\mathbb{R}$ to A [12, (2.1.3), Theorem 3.1.7]. (The superscript $|\ |$ means that the functions are integrable against any power of the absolute value function $|\ |$.) Let $B = L_1^{|\ |}(\mathbb{R}, A, \alpha)$. We show that $\|\ \|_m$ are in fact equivalent to a family of submultiplicative seminorms on which $\hat{\alpha}$ acts isometrically on each seminorm. It is easy to see that the seminorms

$$\|f\|_m' = \int_{\mathbb{R}} (1 + |r|)^m \|f(r)\|_m \, dr,$$

where $f \in B$, are isometric for the action $\hat{\alpha}$ (since $\|e^{2\pi i r s} f(r)\|_m = \|f(r)\|_m$). These norms $\|\ \|_m'$ topologize B, and so are equivalent to the $\|\ \|_m$. Hence

for each $f \in B$, the function $s \mapsto \|\hat{\alpha}_s(f)\|_m$ is bounded by $C\|f\|'_k$ for some sufficiently large $k \in \mathbb{N}$, $C > 0$ chosen independently of f. Then the functions $f \mapsto \sup_{s \in \mathbb{R}} \|\hat{\alpha}_s(f)\|_m$ are easily seen to be a well-defined family of submultiplicative, $\hat{\alpha}$-isometric, seminorms topologizing B. Replace our original seminorms $\| \ \|_m$ with these ones.

The seminorms $\|f\|'_m = \max_{k \leq m} \|D^k f\|_m$ topologize $\mathscr{S}(\mathbb{R}, A, \alpha)$ and are submultiplicative since

$$\|f * g\|'_m = \max_{k \leq m} \|f * D^k g\|_m \leq \|f\|_m \max_{k \leq m} \|D^k g\|_m \leq \|f\|'_m \|g\|'_m,$$

for $f, g \in \mathscr{S}(\mathbb{R}, A, \alpha)$. By the product rule,

$$\|D^k \hat{\alpha}_s(f)\|_m \leq \text{poly}(s) \left(\max_{i \leq k} \|\hat{\alpha}_s D^i f\|_m \right)$$

$$= \text{poly}(s) \left(\max_{i \leq k} \|D^i f\|_m \right),$$

so $\|\hat{\alpha}_s(f)\|'_m \leq \text{poly}(s)\|f\|'_m$ and $\hat{\alpha}$ is m-tempered (see Definition 1.1.3). $\square$

1.2.4 *Remark.* Note that the above proof shows that it is possible to have an m-tempered action of $\mathbb{R}$ on a Fréchet algebra A for which there is no choice of seminorms (submultiplicative or otherwise) for which α acts isometrically. For example, let $A = \mathscr{S}(\mathbb{R})$ with pointwise (convolution) multiplication, and let $\mathbb{R}$ act by translation (dual of translation). Then for any family $\{\| \ \|_m\}$ of seminorms topologizing $\mathscr{S}(\mathbb{R})$, and for any nonzero $f \in \mathscr{S}(\mathbb{R})$, there must be infinitely many m such that $\|\alpha_s(f)\|_m$ is not bounded in s.

We remark that the m-temperedness of $\hat{\alpha}$ is not needed for the m-convexity of the crossed product $\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \alpha), \hat{\alpha})$ by the following lemma (m-temperedness is not required in its proof). However Lemma 1.2.3 will be needed, for example, for the m-convexity of the algebra $F(F(A, \alpha), \widehat{I(\alpha)})$ used below.

1.2.5 **Lemma.** *Let A, α and $\hat{\alpha}$ be as in the previous lemma. Then there is a natural isomorphism (Takesaki-Takai duality)*

$$d_\alpha: \mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \alpha), \hat{\alpha}) \cong \mathscr{K}^\infty \otimes A.$$

Proof. This is proved without the m-convexity conditions in Lemma 2.8 of [7], but with a different definition of $\mathscr{K}^\infty$ than ours. Define $\mathscr{K}_{\mathbb{R}}^\infty = \mathscr{S}(\mathbb{R}^2)$ with matrix multiplication $(f * g)(r, t) = \int_{\mathbb{R}} f(r, s) g(s, t) \, ds$. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be the orthonormal basis of $L_2(\mathbb{R})$ consisting of the Hermite functions. For $f \in \mathscr{S}(\mathbb{R})$ define constants c_n by the formula $f = \sum_{n \in \mathbb{N}} c_n \xi_n$, and define a map $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{N})$ by $f \mapsto \{c_n\}_{n \in \mathbb{N}}$. This gives an inner product preserving isomorphism of $\mathscr{S}(\mathbb{R})$ with $\mathscr{S}(\mathbb{N})$ [11, Theorem V.13]. The tensor product of this map with itself gives an isomorphism $\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{N}) \otimes \mathscr{S}(\mathbb{N})$, which is a Fréchet algebra isomorphism of $\mathscr{K}_{\mathbb{R}}^\infty$ with $\mathscr{K}^\infty$. $\square$

1.2.6 Lemma. *Let α be a smooth, m-tempered action of $\mathbb{R}$ on a Fréchet algebra A. Let τ be the trivial action. Then the following diagram commutes:*

$$
\begin{array}{ccc}
RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes A) & \xrightarrow{\ \mathrm{id} \otimes \theta^{(0)}_{A,\alpha}\ } & RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}, A, \alpha)) \\
{\scriptstyle \mathrm{id}\,\theta^{(0)}_{A,\tau}} \Big\downarrow & & \Big\downarrow {\scriptstyle \theta^{(0)}_{\mathscr{S}(\mathbb{R}, A, \alpha),\hat{\alpha}}} \\
RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}, A, \tau)) & & RK_*(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \alpha), \hat{\alpha}) \\
{\scriptstyle \theta^{(0)}_{\mathscr{S}(\mathbb{R}, A, \tau),\hat{\tau}}} \Big\downarrow & & \Big\downarrow {\scriptstyle (d_\alpha)_*} \\
RK_*(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \tau), \hat{\tau}) & \xrightarrow[\ (d_\tau)_*\ ]{} & RK_*(\mathscr{K}^\infty \otimes A)
\end{array}
$$

Proof. We make the obvious identification of $\mathscr{S}(\mathbb{R}, A, \tau)$ with $\mathscr{S}(\mathbb{R}) \otimes A$. (Since $I_\infty(\tau)$ is the trivial action, $F(A, \tau) \cong \mathscr{S}(\mathbb{R}) \otimes I_\infty A$, and the upper left vertical map is induced by the identity.) Consider the following diagram of Fréchet algebras.

$$
\begin{array}{ccccc}
\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes A & \xleftarrow{\ \mathrm{id}\otimes\varepsilon_0\ } & \mathscr{S}(\mathbb{R}) \otimes F(A, \alpha) & \xrightarrow{\ \mathrm{id}\otimes\varepsilon_1\ } & \mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}, A, \alpha) \\
{\scriptstyle \varepsilon_0} \big\uparrow & & {\scriptstyle \varepsilon_0} \big\uparrow & & {\scriptstyle \varepsilon_0} \big\uparrow \\
F(\mathscr{S}(\mathbb{R}) \otimes A, \hat{\tau}) & \xleftarrow{\ F(\varepsilon_0)\ } & F(F(A, \alpha), \widehat{I(\alpha)}) & \xrightarrow{\ F(\varepsilon_1)\ } & F(\mathscr{S}(\mathbb{R}, A, \alpha), \hat{\alpha}) \\
{\scriptstyle \varepsilon_1} \big\downarrow & & {\scriptstyle \varepsilon_1} \big\downarrow & & {\scriptstyle \varepsilon_1} \big\downarrow \\
\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}) \otimes A, \hat{\tau}) & \xleftarrow{\ \bar{\varepsilon}_0\ } & \mathscr{S}(\mathbb{R}, F(A, \alpha), \widehat{I(\alpha)}) & \xrightarrow{\ \bar{\varepsilon}_1\ } & \mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \alpha), \widehat{\hat{\alpha}}) \\
{\scriptstyle d_\tau} \big\downarrow & & {\scriptstyle D_{I(\alpha)}} \big\downarrow & & {\scriptstyle d_\alpha} \big\downarrow \\
\mathscr{K}^\infty \otimes A & \xleftarrow{\ \mathrm{id}\otimes\mathrm{ev}_0\ } & \mathscr{K}^\infty \otimes I_\infty A & \xrightarrow{\ \mathrm{id}\otimes\mathrm{ev}_1\ } & \mathscr{K}^\infty \otimes A
\end{array}
$$

$(*)$

Here $\bar{\varepsilon}_0$ and $\bar{\varepsilon}_1$ are the maps induced on the crossed products by ε_0 and ε_1.

Naturality of the construction F and of the maps ε_0 and ε_1 implies that the top four squares commute, and naturality of Takesaki-Takai duality gives commutativity for the bottom two squares. All the upward arrows along the top (labelled ε_0) are isomorphisms on K-theory by Corollary 1.1.12. We claim that the same is true for the left arrows on the left side. This is certainly the case for $\mathrm{id}\otimes\varepsilon_0$, since it is the suspension of an isomorphism on K-theory. Therefore $F(\varepsilon_0)_*$ is an isomorphism, because the other three maps in the top left square are isomorphisms on K-theory. Furthermore, ev_0 is an isomorphism on K-theory by Lemma 1.1.10(1), so its stabilization $\mathrm{id}\otimes\mathrm{ev}_0$ is too. Since d_τ and $d_{I(\alpha)}$ are actually algebra isomorphisms, it follows that $(\bar{\varepsilon}_0)_*$ is an isomorphism. This proves the claim.

We now apply RK_* to the entire diagram $(*)$. The K-theory diagram still commutes when all the top vertical arrows and the left horizontal arrows are inverted. Since $\theta^{(0)} = (\varepsilon_1)_* \circ (\varepsilon_0)_*^{-1}$, we obtain the commutative diagram

$$RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes A) \xrightarrow{\ \mathrm{id}\,\otimes\theta^{(0)}_{A,\alpha}\ } RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}, A, \alpha))$$

$$\theta^{(0)}_{\mathscr{S}(\mathbb{R})\otimes A,\,\hat{\tau}}\Big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow\theta^{(0)}_{\mathscr{S}(\mathbb{R},A,\alpha),\,\hat{\alpha}}$$

$$RK_*(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}) \otimes A, \hat{\tau})) \qquad\qquad\qquad RK_*(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \alpha), \hat{\alpha}))$$

$$(d_\tau)_*\Big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow(d_\alpha)_*$$

$$RK_*(\mathscr{K}^\infty \otimes A) \xrightarrow{\ (\mathrm{id}\,\otimes\,\mathrm{ev}_1)_*\,\circ(\mathrm{id}\,\otimes\,\mathrm{ev}_0)_*^{-1}\ } RK_*(\mathscr{K}^\infty \otimes A)$$

The bottom horizontal map is the identity, so this is the diagram in the statement of the lemma. □

1.2.7 Theorem. *Let α be a smooth m-tempered action of $\mathbb{R}$ on a Fréchet algebra A. Then the Thom map $\theta: RK_*(A) \to RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha))$ is an isomorphism.*

Proof. Let τ denote the trivial action of $\mathbb{R}$ on A. Consider Diagram A. The two triangles commute by definition, the square on the middle left commutes by Lemma 1.2.2, and the rectangle at the bottom commutes by Lemma 1.2.6. Thus Diagram A commutes.

We now claim that the map from $RK_*(A)$ to $RK_*(\mathscr{K}^\infty \otimes A)$ is the usual stabilization isomorphism. We will prove this by going along the left side and the bottom, and reducing the problem to Theorem 4.5 of [8]. Using Bott periodicity it suffices to prove this for RK_0. Let $\eta \in RK_0(A)$, and choose a representative idempotent $p \in B = M_2((\mathscr{K}^\infty \otimes A)^+)$. Note that $p - \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \in M_2(\mathscr{K}^\infty \otimes A)$. Define $\varphi: \mathbb{C} \to M_2((\mathscr{K}^\infty \otimes A)^+)$ by $\varphi(\lambda) = \lambda p$. Then we have Diagram B, which is also commutative. Note that the composition of the horizontal maps on the top is precisely the map from $RK_*(A)$ to $RK_*(\mathscr{K}^\infty \otimes A)$ along the left and bottom of Diagram A. The top vertical arrows are induced by the map $RK_0(A) \cong RK_0(M_2(\mathscr{K}^\infty \otimes A)) \to RK_0(B)$ coming from $\psi: M_2(\mathscr{K}^\infty \otimes A) \hookrightarrow B$. The bottom vertical arrows are induced by inclusions in the corresponding C^*-algebras, and the groups along the bottom row are the conventional K-theory of C^*-algebras. The map θ^* in the bottom row of Diagram B is constructed in the same manner as $\theta^{(0)}$. That is, let $D = C_0(\mathbb{R}^2)$, let ev_0, $\mathrm{ev}_1: C([0, 1]) \otimes D \to D$ be the evaluation maps, and let ε_0 and ε_1 be the crossed products of ev_0 and ev_1 by $\mathbb{R}$; then $\theta^* = (\varepsilon_1)_* \circ (\varepsilon_0)_*^{-1}$. It is much easier to prove that $(\varepsilon_0)_*$ is invertible in the C^*-algebra case, since one can use $C_0((0, 1]) \otimes D$ in place of $C_\infty D$. It is also trivial to check that θ^* is the same map as in Theorem 4.5 of [8].

It follows from this same theorem that the map $K_0(\mathbb{C}) \to K_0(\mathscr{K})$ along the bottom row is the usual stabilization map. Now the maps $RK_0(\mathbb{C}) \to K_0(\mathbb{C})$ and $RK_0(\mathscr{K}^\infty) \to K_0(\mathscr{K})$ are isomorphisms, the range of the map φ_* contains $[p] = \varphi_*([1])$, the maps ψ_* and $(\mathrm{id}\,\otimes\psi)_*$ are injective since they come from split exact sequences, and $[p] = \psi_*([p])$ is in the range of ψ_*. Therefore the map $RK_0(A) \to RK_0(\mathscr{K}^\infty \otimes A)$ across the top row of Diagram B sends $\eta = [p]$ to its image under the usual stabilization map. Since $\eta \in RK_0(A)$ was arbitrary, this proves the claim.

It follows that $\theta_{\mathscr{S}(\mathbb{R}, A, \alpha)} \circ \theta_A$ in Diagram A is an isomorphism. Applying this to the action $\hat{\alpha}$ of $\mathbb{R}$ on $\mathscr{S}(\mathbb{R}, A, \alpha)$ now shows that $\theta_{\mathscr{S}(\mathbb{R}, A, \alpha)}$, and hence also θ_A, is an isomorphism. □
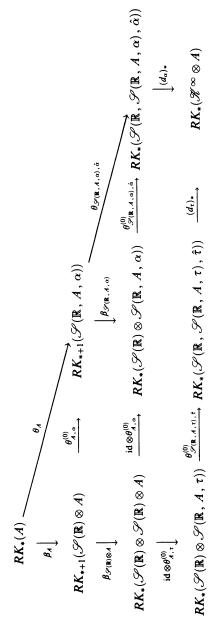
DIAGRAM A

$$RK_*(A)$$

$$\beta_A \downarrow \qquad\qquad \xrightarrow{\ \theta_A\ }$$

$$RK_{*+1}(\mathscr{S}(\mathbb{R}) \otimes A) \qquad\qquad RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha))$$

$$\beta_{\mathscr{S}(\mathbb{R})\otimes A} \downarrow \qquad\qquad \downarrow \beta_{\mathscr{S}(\mathbb{R}, A, \alpha)}$$

$$RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes A) \xrightarrow{\ \theta^{(0)}_{A,\alpha}\ } RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}, A, \alpha))$$

$$\mathrm{id}\otimes\theta^{(0)}_{A,\alpha} \downarrow$$

$$\xrightarrow{\ \theta^{(0)}_{\mathscr{S}(\mathbb{R}, A, \alpha), \hat\alpha}\ } RK_*(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \alpha), \hat\alpha))$$

$$\theta_{\mathscr{S}(\mathbb{R}, A, \alpha), \hat\alpha} \nearrow$$

$$\downarrow (d_\alpha)_*$$

$$RK_*(\mathscr{H}^\infty \otimes A)$$

$$\mathrm{id}\otimes\theta^{(0)}_{A,\tau} \downarrow$$

$$RK_*(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}, A, \tau)) \xrightarrow{\ \theta^{(0)}_{\mathscr{S}(\mathbb{R}, A, \tau), \hat\tau}\ } RK_*(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}, A, \tau), \hat\tau)) \xrightarrow{\ (d_\tau)_*\ }$$

DIAGRAM B

$$RK_0(A) \xrightarrow{\beta\circ\beta} RK_0(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes A) \xrightarrow{\theta^{(0)}} RK_0(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}) \otimes A, \hat\tau)) \xrightarrow{(d_\tau)_*} RK_0(\mathscr{H}^\infty \otimes A)$$

$$\psi_* \downarrow \qquad (\mathrm{id}\otimes\mathrm{id}\otimes\psi)_* \downarrow \qquad\qquad\qquad\qquad (\mathrm{id}\otimes\psi)_* \downarrow$$

$$RK_0(B) \xrightarrow{\beta\circ\beta} RK_0(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes B) \xrightarrow{\theta^{(0)}} RK_0(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}) \otimes B, \hat\tau)) \xrightarrow{(d_\tau)_*} RK_0(\mathscr{H}^\infty \otimes B)$$

$$\varphi_* \uparrow \qquad (\mathrm{id}\otimes\mathrm{id}\otimes\varphi)_* \uparrow \qquad\qquad\qquad\qquad (\mathrm{id}\otimes\varphi)_* \uparrow$$

$$RK_0(\mathbb{C}) \xrightarrow{\beta\circ\beta} RK_0(\mathscr{S}(\mathbb{R}) \otimes \mathscr{S}(\mathbb{R}) \otimes \mathbb{C}) \xrightarrow{\theta^{(0)}} RK_0(\mathscr{S}(\mathbb{R}, \mathscr{S}(\mathbb{R}), \hat\tau)) \xrightarrow{(d_\tau)_*} RK_0(\mathscr{H}^\infty)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$K_0(\mathbb{C}) \xrightarrow{\beta\circ\beta} K_0(C_0(\mathbb{R}^2)) \xrightarrow{\theta^*} K_0(C^*(\mathbb{R}, C_0(\mathbb{R}), \hat\tau)) \longrightarrow K_0(\mathscr{K})$$

1.3   Replacing A^∞ with A and $\mathscr{S}(\mathbb{R})$ with $L_1(\mathbb{R})$. We show that the smooth Thom isomorphism still holds if the action of α on A is strongly continuous but not smooth. We also show that if α leaves each seminorm on A invariant, then the Schwartz algebra $\mathscr{S}(\mathbb{R}, A, \alpha)$ may be replaced by $L_1(\mathbb{R}, A, \alpha)$.

1.3.1   Theorem. *Let A be a Fréchet algebra with m-tempered and strongly continuous action α of $\mathbb{R}$. Let A^∞ denote the Fréchet algebra of C^∞-vectors. Then the inclusion map $\mathscr{S}(\mathbb{R}, A^\infty, \alpha) \hookrightarrow \mathscr{S}(\mathbb{R}, A, \alpha)$ induces an isomorphism on K-theory.*

Proof. If $f \in \mathscr{S}(\mathbb{R}, A, \alpha)$, define $\alpha_s(f)(r) = \alpha_s(f(r))$. Then α defines an action of $\mathbb{R}$ on $\mathscr{S}(\mathbb{R}, A, \alpha)$ by *algebra* automorphisms. Viewing $\mathscr{S}(\mathbb{R}, A, \alpha)$ as the Fréchet space $\mathscr{S}(\mathbb{R}) \otimes A$, note that α is just the tensor product of the identity map on $\mathscr{S}(\mathbb{R})$ with the original action of α on A. The strong continuity of α on $\mathscr{S}(\mathbb{R}, A, \alpha)$ follows easily from the strong continuity of α on A. The set of C^∞-vectors for the action of α on $\mathscr{S}(\mathbb{R}, A, \alpha)$ is easily seen to be $\mathscr{S}(\mathbb{R}, A^\infty, \alpha)$. (For example, use [12, Theorem A.8] and the nuclearity of $\mathscr{S}(\mathbb{R})$.)

We show that α is an m-tempered action on $\mathbb{R}$ on $\mathscr{S}(\mathbb{R}, A, \alpha)$. To do this, we will show that the condition for an m-tempered action introduced in Theorem 3.1.18 of [12] is satisfied. Since the action of α on A is m-tempered by assumption, we let $\|\ \|_k$ be an increasing family of submultiplicative seminorms topologizing A, satisfying the conditions of Definition 1.1.3. For fixed k, let $C > 0$ and $d \in \mathbb{N}$ be such that

$$(*) \qquad \|\alpha_s(a)\|_k \leq C\omega(s)^d \|a\|_k, \qquad s \in \mathbb{R}, \ a \in A,$$

where $\omega(s) = 1 + |s|$. Define seminorms (not necessarily submultiplicative) for $\mathscr{S}(\mathbb{R}, A, \alpha)$ by

$$\|f\|_{m,l,k} = \int_\mathbb{R} \omega(r)^m \|f^{(l)}(r)\|_k \, dr, \qquad f \in \mathscr{S}(\mathbb{R}, A, \alpha).$$

Let $s_1, \ldots, s_n \in \mathbb{R}$ and $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}, A, \alpha)$. In the following computation, we let $r_n = r - r_1 - \cdots - r_{n-1}$, $\eta_1 = 0$, and $\eta_k = r_1 + \cdots + r_{k-1}$. Then

$$(**)$$
$$\|\alpha_{s_1}(f_1) \cdots \alpha_{s_n}(f_n)\|_{m,l,k}$$
$$= \int_\mathbb{R} \omega(r)^m \left\| \int_\mathbb{R} \cdots \int_\mathbb{R} \alpha_{s_1 + \eta_1}(f_1(r_1)) \right.$$
$$\left. \cdots \alpha_{s_{n-1} + \eta_{n-1}}(f_{n-1}(r_{n-1})) \alpha_{s_n + \eta_n}(f_n^{(l)}(r_n)) \, dr_1 \cdots dr_{n-1} \right\|_k dr$$
$$\leq \int_\mathbb{R} \cdots \int_\mathbb{R} \omega(r)^m \left\| \alpha_{s_1 + \eta_1}(f_1(r_1)) \cdots \right.$$
$$\left. \alpha_{s_{n-1} + \eta_{n-1}}(f_{n-1}(r_{n-1})) \alpha_{s_n + \eta_n}(f_n^{(l)}(r_n)) \right\|_k dr_1 \cdots dr_n.$$

We bound the normed expression in the integrand. We use $(*)$ and $\eta_1 = 0$ and $\eta_2 = r_1$ in the first step, and proceed analogously through the remaining steps.

$$\|\alpha_{s_1+\eta_1}(f_1(r_1))\cdots\alpha_{s_{n-1}+\eta_{n-1}}(f_{n-1}(r_{n-1}))\alpha_{s_n+\eta_n}(f_n^{(l)}(r_n))\|_k$$

$$\leq C\omega(s_1)^d\|f_1(r_1)\|_k\|\alpha_{s_2-s_1+r_1}(f_2(r_2))\cdots$$

$$\alpha_{s_{n-1}-s_1+\eta_{n-1}}(f_{n-1}(r_{n-1}))\alpha_{s_n-s_1+\eta_n}(f_n^{(l)}(r_n))\|_k$$

$$\leq C^2(\omega(s_1)\omega(s_2-s_1)\omega(r_1))^d\|f_1(r_1)\|_k\|f_2(r_2)\|_k$$

$$\cdot\|\alpha_{s_3-s_2+r_2}(f_3(r_3))\cdots\alpha_{s_{n-1}-s_2+\eta_{n-1}}(f_{n-1}(r_{n-1}))\alpha_{s_n-s_2+\eta_n}(f_n^{(l)}(r_n))\|_k$$

$$\leq C^n(\omega(s_1)\cdots\omega(s_n-s_{n-1}))^d(\omega(r_1)\cdots\omega(r_n))^d\|f_1(r_1)\|_k$$

$$\cdots\|f_{n-1}(r_{n-1})\|_k\|f_n^{(l)}(r_n)\|_k.$$

Hence $(**)$ is bounded by

$$C^n(\omega(s_1)\omega(s_2-s_1)\cdots\omega(s_n-s_{n-1}))^d\|f_1\|_{m+d,0,k}\cdots\|f_{n-1}\|_{m+d,0,k}\|f_n\|_{m+d,l,k}.$$

Thus the increasing seminorms $\|f\|_k' = \max_{l\leq k}\int_{\mathbb{R}}\omega(r)^k\|f^{(l)}(r)\|_k\,dr$ on $\mathscr{S}(\mathbb{R}, A, \alpha)$ satisfy the condition

$$\|\alpha_{s_1}(f_1)\cdots\alpha_{s_n}(f_n)\|_k' \leq C^n(\omega(s_1)\omega(s_2-s_1)\cdots\omega(s_n-s_{n-1}))^d\|f_1\|_{k+d}'\cdots\|f_n\|_{k+d}'$$

of Theorem 3.1.18 of [12]. It follows that α is an m-tempered action on $\mathscr{S}(\mathbb{R}, A, \alpha)$.

Now let $\|\ \|_n$ be seminorms topologizing $\mathscr{S}(\mathbb{R}, A, \alpha)$ as in Definition 1.1.3. Let B_n be the completion of $\mathscr{S}(\mathbb{R}, A, \alpha)$ in $\|\ \|_n$, and let $\alpha^{(n)}$ denote the extension of α to B_n. Then B_n is a Banach algebra and $\alpha^{(n)}$ is a strongly continuous action of $\mathbb{R}$ by algebra automorphisms. Let $B_{n,m}$ denote the Banach algebra of m-times differentiable vectors for the action $\alpha^{(n)}$ on B_n. Then $B_{n,n}$ is dense and spectrally invariant in B_n. (For example, this follows from the estimates in [13, Theorem 2.2].) Since $\mathscr{S}(\mathbb{R}, A, \alpha) = \varprojlim B_n$, we have

$$\mathscr{S}(\mathbb{R}, A^\infty, \alpha) = \mathscr{S}(\mathbb{R}, A, \alpha)^\infty = (\varprojlim B_n)^\infty$$

$$= \varprojlim B_n^\infty = \varprojlim_n\varprojlim_m B_{n,m} = \varprojlim B_{n,n}.$$

A direct application of Lemma 1.1.9(2) now tells us that the inclusion

$$\mathscr{S}(\mathbb{R}, A^\infty, \alpha) \hookrightarrow \mathscr{S}(\mathbb{R}, A, \alpha)$$

is an isomorphism on K-theory. $\square$

Corollary 1.3.2. *Let A be a Fréchet algebra with m-tempered, strongly continuous (not necessarily smooth) action α of $\mathbb{R}$. Then there is an isomorphism*

$$RK_*(A) \cong RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha)).$$

Proof. The argument in the last paragraph of the proof of Theorem 1.3.1 shows that the inclusion $A^\infty \hookrightarrow A$ is an isomorphism on K-theory. We have the diagram

$$\begin{array}{ccc} RK_*(A^\infty) & \xrightarrow{\ i_*\ } & RK_*(A) \\ {\scriptstyle\theta}\downarrow & & \\ RK_{*+1}(\mathscr{S}(\mathbb{R}, A^\infty, \alpha)) & \xrightarrow{\ i_*\ } & RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha)), \end{array}$$

where the bottom arrow is an isomorphism by Theorem 1.3.1, and θ is the Thom map from §1. The composition of the three isomorphisms gives the desired isomorphism on the right-hand side. □

Definition 1.3.3. We say that α acts *isometrically* on a Fréchet algebra A if there exists a family $\{\| \ \|_k\}$ of submultiplicative seminorms topologizing A such that α acts isometrically for each $\| \ \|_k$. (This is equivalent to the existence of not necessarily submultiplicative seminorms $\| \ \|_k$ such that $\|\alpha_s(a)\|_k$ is a bounded function of $s \in \mathbb{R}$ for each $a \in A$ and k. See the proof of Lemma 1.2.3 and Remark 1.2.4.) Define

$$L_1(\mathbb{R}, A, \alpha) = \left\{ f \colon \mathbb{R} \to A \,\middle|\, f \text{ measurable and} \right.$$

$$\left. \int_{\mathbb{R}} \|f(r)\|_k \, dr < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

(See [12, §2] for details.)

If α is an isometric action on A, then $L_1(\mathbb{R}, A, \alpha)$ is a Fréchet algebra under convolution. However, if α is not isometric, then $L_1(\mathbb{R}, A, \alpha)$ is unlikely to be an algebra. Note that if α is isometric and A is a Banach algebra, then so is $L_1(\mathbb{R}, A, \alpha)$.

Theorem 1.3.4. *Let A be a Fréchet algebra with isometric action α of $\mathbb{R}$. Then the inclusion map $\mathscr{S}(\mathbb{R}, A, \alpha) \hookrightarrow L_1(\mathbb{R}, A, \alpha)$ is an isomorphism on K-theory.*

Proof. Let $\| \ \|_k$ be submultiplicative α-isometric seminorms for A. Let A_k be the completion of $A/\operatorname{Ker}(\| \ \|_k)$ in $\| \ \|_k$, and let $B_k = L_1(\mathbb{R}, A_k, \alpha)$. Then the inverse limit $\varprojlim B_k$ is equal to $L_1(\mathbb{R}, A, \alpha)$. Define

$$\|f\|'_{n,k} = \sum_{i+j=n} \int_{\mathbb{R}} \omega(r)^i \|f^{(j)}(r)\|_k \, dr, \qquad f \in \mathscr{S}(\mathbb{R}, A, \alpha),$$

where $\omega(r) = 1 + |r|$. Let $B_{n,k}$ be the completion of $\mathscr{S}(\mathbb{R}, A, \alpha)/\operatorname{Ker}(\| \ \|'_{n,k})$ in $\| \ \|'_{n,k}$. The seminorms $\| \ \|'_{n,k}$ are submultiplicative under convolution since the action is isometric. Moreover, omitting the subscript k everywhere for convenience, we have

$$\|f * g\|'_n = \sum_{i+j=n} \int_{\mathbb{R}} \omega(r)^i \left\| \int_{\mathbb{R}} f(t)\alpha_t(g^{(j)}(r-t)) \, dt \right\| dr$$

$$\leq \sum_{i+j=n} \int_{\mathbb{R}} \omega(r)^i \int_{\mathbb{R}} \|f(t)\| \, \|g^{(j)}(r-t))\| dt \, dr$$

$$\leq 2^n \sum_{i+j=n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \omega(t)^i \|f(t)\| \, \|g^{(j)}(r-t)\| \, dt \, dr \right.$$

$$\left. + \int_{\mathbb{R}} \int_{\mathbb{R}} \|f(t)\| \omega(r-t)^i \|g^{(j)}(r-t)\| \, dt \, dr \right)$$

$$\leq 2^n \left(\sum_{i_1+i_2+j_1+j_2=n} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(t)^{i_1} \|f^{(i_2)}(t)\| \omega(r-t)^{j_1} \|g^{(j_2)}(r-t)\| \, dr \, dt \right)$$
$$+ 2^n \|f\|_0' \|g\|_n'$$
$$\leq 2^{n+1} \sum_{i+j=n} \|f\|_i' \|g\|_j' \, .$$

This condition implies the spectral invariance of $B_{n,k}$ in B_k for every n [13, (1.11), Theorem 1.17]. (The main point here is that the sum on the right-hand side is over $i+j=n$.) In particular, $B_{k,k}$ is spectral invariant in B_k. We have $\mathscr{S}(\mathbb{R}, A, \alpha) = \varprojlim_k \varprojlim_n B_{n,k} = \varprojlim B_{k,k}$, so by Lemma 1.1.9(2), we have that $\mathscr{S}(\mathbb{R}, A, \alpha) \hookrightarrow L_1(\mathbb{R}, A, \alpha)$ is an isomorphism on K-theory. $\square$

Corollary 1.3.5. *Let A be a Fréchet algebra with isometric and strongly continuous action α of $\mathbb{R}$. Then there is an isomorphism*

$$RK_*(A) \cong RK_{*+1}(L_1(\mathbb{R}, A, \alpha)) \, .$$

If A is a Banach algebra, then so is $L_1(\mathbb{R}, A, \alpha)$, and RK may be replaced with K in the isomorphism.

Proof. This is similar to Corollary 1.3.2. By Theorem 1.3.4 and Corollary 1.3.2 we have isomorphisms

$$RK_*(A) \xrightarrow{\theta} RK_{*+1}(\mathscr{S}(\mathbb{R}, A, \alpha)) \xrightarrow{i_*} RK_{*+1}(L_1(\mathbb{R}, A, \alpha)) \, .$$

Thus by taking the composition, we get the desired isomorphism. For replacing RK with K, apply [10, Corollary 7.8]. $\square$

2. Smooth crossed products by $\mathbb{Z}$

We introduce the smooth mapping torus, and use it and the smooth Thom isomorphism of §1 to obtain a smooth version of the Pimsner-Voiculescu exact sequence. This is an imitation of the proof of the Pimsner-Voiculescu exact sequence for C^*-algebras in [1, §10.3-4].

2.1 Definition. Let α be an automorphism of the algebra A. Define the *mapping torus* $M(A, \alpha)$ to be

$$M(A, \alpha) = \{f \colon [0, 1] \to A, \ f \text{ continuous } | f(1) = \alpha(f(0))\} \, .$$

Then $M(A, \alpha)$ is a Fréchet algebra under pointwise multiplication, with the topology of uniform convergence. Define the *smooth mapping torus* $M_\infty(A, \alpha)$ to be the set of functions

$$M_\infty(A, \alpha) = \{f \colon [0, 1] \to A, \ f \text{ differentiable } | f^{(l)}(1) = \alpha(f^{(l)}(0)), \ l \in \mathbb{N}\} \, .$$

We topologize $M_\infty(A, \alpha)$ by uniform convergence of derivatives. Under pointwise multiplication, the sum of the sup norms of the first n derivatives is submultiplicative modulo constants. Hence $M_\infty(A, \alpha)$ is a Fréchet algebra.

We define the *smooth α-suspension* $S_\infty(A, \alpha)$ to be the closed subalgebra of $M_\infty(A, \alpha)$ given by

$$S_\infty(A, \alpha) = \{f \colon [0, 1] \to A, \ f \text{ differentiable } | f(1) = f(0) = 0,$$
$$f^{(l)}(1) = \alpha(f^{(l)}(0)), \ l \in \mathbb{N}\} \, .$$

2.2 Lemma. *If A is a Fréchet algebra with automorphism α, and $SA = \{f \in C(\mathbb{T}, A)|f(0) = 0\}$ is the continuous suspension of A, then the inclusion $S_\infty(A, \alpha) \hookrightarrow SA$ is an isomorphism on K-theory. In particular, there is a natural isomorphism $RK_*(A) \cong RK_{*+1}(S_\infty(A, \alpha))$.*

Proof. The proof is similar to that of Lemma 1.1.10(3). We use $[0, 1]$ in place of $\mathbb{R}$, and replace the rapid decay conditions on the derivatives with the condition that the derivatives match correctly at 0 and 1, as in Definition 2.1. The Fourier transform argument becomes unnecessary. Details are omitted. $\square$

2.3 Lemma. *Let A be a Fréchet algebra, and let α be an automorphism of A. Then there is a natural six term exact sequence*

$$RK_1(A) \xrightarrow{\delta_0} RK_0(M_\infty(A, \alpha)) \longrightarrow RK_0(A)$$

$$\text{id} - (\alpha^{-1})_* \Big\uparrow \qquad\qquad\qquad\qquad \Big\downarrow \text{id} - (\alpha^{-1})_*$$

$$RK_1(A) \longleftarrow RK_1(M_\infty(A, \alpha)) \xleftarrow{\delta_1} RK_0(A).$$

Here, δ_i is the isomorphism of Lemma 2.2, followed by the inclusion of $S_\infty(A, \alpha)$ in $M_\infty(A, \alpha)$.

Proof. This is essentially Proposition 10.4.1 of [1]. We give a different proof, since the one in [1] uses a formula for the connecting homomorphisms which has not been proved in representable K-theory.

We will show that the sequence of the lemma is the one associated to the short exact sequence

$$(*) \qquad\qquad 0 \to S_\infty(A, \alpha) \to M_\infty(A, \alpha) \xrightarrow{\text{ev}_1} A \to 0.$$

The only thing requiring proof is the identification of the vertical maps $\text{id} - (\alpha^{-1})_*$, and it suffices to do the one on the left.

We replace $(*)$ by

$$(**) \qquad\qquad 0 \to SA \to M(A, \alpha) \xrightarrow{\text{ev}_1} A \to 0.$$

The inclusion maps $S_\infty(A, \alpha) \to SA$ and $A \to A$ are isomorphisms on K-theory. Therefore the Five Lemma implies that $M_\infty(A, \alpha) \to M(A, \alpha)$ is too. This justifies the replacement.

Let $\beta: RK_1(A) \to RK_0(SA)$ be Bott periodicity, and let $\partial: RK_1(A) \to RK_0(SA)$ be the connecting homomorphism from $(**)$. We show that $\beta^{-1} \circ \partial = \text{id} - (\alpha^{-1})_*$. Using the construction of the boundary map in Proposition 3.11 and Remark 3.12 of [10], we identify $\beta^{-1} \circ \partial$ with the composition

$$RK_1(A) \xrightarrow{\beta} RK_0(SA) \xrightarrow{\varphi_*^{-1} \circ \nu_*} RK_0(SA) \xrightarrow{\beta^{-1}} RK_1(A).$$

Following Remark 3.12 of [10], and being careful of the conflicting notation, we obtain $\varphi: SA \to D$ and $\nu: SA \to D$ as

$$D = \{(g, f) \in C([0, 1], A) \oplus C([0, 1], A)|$$
$$g(1) = \alpha(g(0)), f(0) = g(1), \text{ and } f(1) = 0\},$$
$$\varphi(g) = (g, 0), \quad \text{and} \quad \nu(f) = (0, f).$$

Then ν is homotopic to the map $\mu: SA \to D$ given by $\mu(f) = (g, 0)$, with $g(t/2) = \alpha^{-1}(f(1-t))$ and $g((t+1)/2) = f(t)$ for $t \in [0, 1]$. A homotopy is given by $\nu_s(f) = (g_s, h_s)$, with

$$h_s(t) = \begin{cases} f(t+s), & t+s \leq 1, \\ 0, & t+s > 1, \end{cases}$$

$$g_s(t/2) = \begin{cases} \alpha^{-1}(f(s-t)), & t+(1-s) \leq 1, \\ 0, & t+(1-s) > 1, \end{cases}$$

$$g_s(t/2 + 1/2) = \begin{cases} f(s+t-1), & (1-t)+(1-s) \leq 1, \\ 0, & (1-t)+(1-s) > 1, \end{cases}$$

where $s \in [0, 1]$ and $t \in [0, 1]$. So $\varphi_*^{-1} \circ \nu_* = \varphi_*^{-1} \circ \mu_*$. Since φ is an isomorphism onto its image, and since φ_* is an isomorphism, we can replace D by SA and $\varphi_*^{-1} \circ \mu_*$ by $(\varphi^{-1} \circ \mu)_*$. This map is $-(S\alpha^{-1})_* + \mathrm{id}$ by Lemma 6.2 of [10]. Therefore $\beta^{-1} \circ \partial = \mathrm{id} - (\alpha^{-1})_*$ by naturality of β. $\square$

If α is a continuous action of the circle group $\mathbb{T}$ on A (automatically tempered and m-tempered since $\mathbb{T}$ is compact—see Definition 1.1.3 and [12, Definition 2.2.4]), we define the Fréchet algebra $\mathscr{S}(\mathbb{T}, A, \alpha)$ to be $C^\infty(\mathbb{T}) \otimes A$ with a twisted convolution product. The following proposition and proof also hold for non m-convex Fréchet algebras A.

2.4 Proposition. *Let $\mathbb{R}$ act on a Fréchet algebra A with m-tempered action γ, and assume that the restriction of γ to $\mathbb{Z}$ is the trivial action. Then γ is m-tempered and factors to an action $\bar\gamma$ of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on A, and $\beta = (\hat{\bar\gamma})_1$ is an automorphism of $\mathscr{S}(\mathbb{T}, A, \bar\gamma)$. The smooth crossed product $\mathscr{S}(\mathbb{R}, A, \gamma)$ is isomorphic to the Fréchet algebra $M_\infty(\mathscr{S}(\mathbb{T}, A, \bar\gamma), \beta)$.*

Proof. The first statement is trivial. In the proof of the second, we use the convention $e(t) = e^{2\pi i t}$. Let $B = \mathscr{S}(\mathbb{R}, A, \gamma)$ and $C = M_\infty(S(\mathbb{T}, A, \bar\gamma), \beta)$. We define a map $\Psi: B \to C$ by

$$\Psi(f)(s)(t+\mathbb{Z}) = \sum_{k \in \mathbb{Z}} e(s(t+k))f(t+k), \qquad s \in [0, 1], \ t \in \mathbb{R}.$$

Since f is a Schwartz function, it follows that $\Psi(f)(s)(t+\mathbb{Z})$ is infinitely differentiable in both variables. Also, the functions $t \mapsto \Psi(f)^{(l)}(s)(t+\mathbb{Z})$ and their derivatives are periodic in t, so $\Psi(f)^{(l)}(s)$ does in fact define a C^∞ function from $\mathbb{T}$ to A for each $s \in [0, 1]$ and $l \in \mathbb{N}$. (We denote the lth derivative in s by $\Psi(f)^{(l)}(s)(t+\mathbb{Z})$ and we denote the lth derivative in t by $\Psi(f)(s)^{(l)}(t+\mathbb{Z})$.) A quick calculation shows that

$$\Psi(f)^{(l)}(1)(t+\mathbb{Z}) = e(t)\Psi(f)^{(l)}(0)(t+\mathbb{Z}) = \beta(\Psi(f)^{(l)}(0))(t+\mathbb{Z}).$$

It follows that $\Psi(f) \in C$. Moreover, it is not to hard to check that if f_n converges to f in the Schwartz topology of B, then the derivatives of $\Psi(f_n)$ converge uniformly to the derivatives of $\Psi(f)$. So Ψ is a continuous linear map of Fréchet spaces.

We verify that Ψ is an algebra homomorphism.

$$\Psi(f * g)(s)(t + \mathbb{Z}) = \sum_{k \in \mathbb{Z}} e(s(t + k)) \int_{\mathbb{R}} f(r) \gamma_r(g(t + k - r)) \, dr$$

$$= \sum_{k, l \in \mathbb{Z}} \int_0^1 e(s(w + l)) f(w + l) e(s(t + k - w)) \gamma_w(g(t + k - w)) \, dw$$

$$= (\Psi(f) * \Psi(g))(s)(t + \mathbb{Z}).$$

We construct an inverse for Ψ. Define a map $\Phi: C \to B$ by

$$\Phi(\xi)(r) = \int_0^1 e(-wr) \xi(w)(r + \mathbb{Z}) \, dw, \qquad r \in \mathbb{R}.$$

We show that $\Phi(\xi)$ lies in B. First note that by integration by parts, we have

$$(*) \qquad (2\pi i r)^p \int_0^1 e(-wr) \xi(w)(r + \mathbb{Z}) \, dw = \int_0^1 e(-wr) \xi^{(p)}(w)(r + \mathbb{Z}) \, dw.$$

Here the boundary condition $\xi^{(k)}(1)(r + \mathbb{Z}) = e(r) \xi^{(k)}(0)(r + \mathbb{Z})$ is essential for eliminating terms at the endpoints. From $(*)$, it follows that

$$\left\| \left(\frac{d}{dr} \right)^l r^p \Phi(\xi)(r) \right\|_m \leq \left(\frac{1}{2\pi} \right)^p \int_0^1 \left\| \left(\frac{d}{dr} \right)^l e(-wr) \xi^{(p)}(w)(r + \mathbb{Z}) \right\|_m dw$$

for each continuous seminorm $\| \ \|_m$ on A. Since the integrand is a bounded continuous function, we see that $\sup_{r \in \mathbb{R}} \|(d/dr)^l r^p \Phi(\xi)(r)\|_m < \infty$ for any l, p, $m \in \mathbb{N}$. Using the commutation relation $[d/dr, r] = 1$ repeatedly, we see that the seminorms $\|\Phi(\xi)\|_{p,l,m} = \sup_{r \in \mathbb{R}} \|r^p \Phi(\xi)^{(l)}(r)\|_m$ on the smooth crossed product B are all finite. Hence $\Phi(\xi) \in B$.

For $\xi \in C$, we have

$$\Psi \circ \Phi(\xi)(s)(t + \mathbb{Z}) = \sum_{k \in \mathbb{Z}} \int_0^1 e(s(t + k)) e(-w(t + k)) \xi(w)(t + k + \mathbb{Z}) \, dw$$

$$= e(st) \sum_{k \in \mathbb{Z}} e(sk) \int_0^1 e(-wk)[e(-wt) \xi(w)(t + \mathbb{Z})] \, dw$$

$$= e(st)[e(-st) \xi(s)(t + \mathbb{Z})] = \xi(s)(t + \mathbb{Z}).$$

The third step uses the fact that the term in brackets is a smooth periodic function of w, and is therefore the sum of its Fourier series. Similarly, one checks that $\Phi \circ \Psi(f) = f$ for $f \in B$. Thus, by the open mapping theorem, Ψ is an isomorphism of Fréchet algebras. $\square$

For an m-tempered action α of $\mathbb{Z}$ on a Fréchet algebra A, we define $\mathscr{S}(\mathbb{Z}, A, \alpha)$ to be $\mathscr{S}(\mathbb{Z}) \otimes A$ with twisted convolution multiplication. Then $\mathscr{S}(\mathbb{Z}, A, \alpha)$ is a Fréchet algebra, by [12, Theorem 3.1.7]. We define the dual action $\hat{\alpha}$ of $\mathbb{T}$ on $\mathscr{S}(\mathbb{Z}, A, \alpha)$ by $\hat{\alpha}_z(f)(n) = e^{2\pi i z n} f(n)$.

2.5 Lemma. *There is a natural isomorphism (Takesaki-Takai duality)*

$$d_\alpha: \mathscr{S}(\mathbb{T}, \mathscr{S}(\mathbb{Z}, A, \alpha), \hat{\alpha}) \to \mathscr{K}^\infty \otimes A.$$

This isomorphism is equivariant for the second dual action $\hat{\hat{\alpha}}_n(f)(z, m) = e^{2\pi i n z} f(z, m)$ on the left and the action on the right generated by the automorphism $e_{jk} \otimes a \mapsto e_{j-1, k-1} \otimes \alpha_1(a)$.

Proof. The proof is essentially the same as for $\mathbb{R}$ in place of $\mathbb{Z}$. We identify $\mathscr{S}(\mathbb{T}, \mathscr{S}(\mathbb{Z}, A, \alpha), \hat{\alpha})$ with the Fréchet space $C^\infty(\mathbb{T}) \otimes \mathscr{S}(\mathbb{Z}) \otimes A$. Then we define d_α to be the composite $\Gamma \circ \mathscr{F}$, where $\mathscr{F}$ is the Fourier transform in the first variable, $\mathscr{F}(f)(k, n) = \int_{\mathbb{T}} e^{2\pi i u k} f(u, n) \, du$ for $n, k \in \mathbb{Z}$, and $\Gamma(\eta)(k, n) = \alpha_{-k}(\eta(n, k - n))$. One checks that d_α transforms the multiplication on $C^\infty(\mathbb{T}) \otimes \mathscr{S}(\mathbb{Z}) \otimes A$ into the usual matrix multiplication on $\mathscr{S}(\mathbb{Z}) \otimes \mathscr{S}(\mathbb{Z}) \otimes A \cong \mathscr{K}^\infty \otimes A$, and transforms the second dual action as described in the lemma. $\square$

2.6 Theorem (Pimsner-Voiculescu exact sequence). *Let α be an m-tempered action of $\mathbb{Z}$ on a Fréchet algebra A. Then we have the following natural six term exact sequence in K-theory.*

$$
\begin{array}{ccccc}
RK_0(A) & \xrightarrow{\ \mathrm{id} - (\alpha_{-1})_* \ } & RK_0(A) & \xrightarrow{\ \iota_* \ } & RK_0(\mathscr{S}(\mathbb{Z}, A, \alpha)) \\[2pt]
\big\uparrow{\scriptstyle \partial} & & & & \big\downarrow{\scriptstyle \partial} \\[2pt]
RK_1(\mathscr{S}(\mathbb{Z}, A, \alpha)) & \xleftarrow{\ \iota_* \ } & RK_1(A) & \xleftarrow{\ \mathrm{id} - (\alpha_{-1})_* \ } & RK_1(A),
\end{array}
$$

where $\iota: A \to \mathscr{S}(\mathbb{Z}, A, \alpha)$ is the inclusion.

Proof. Let $\overline{\gamma} = \hat{\alpha}$ be the dual action of $\mathbb{T}$ on $B = \mathscr{S}(\mathbb{Z}, A, \alpha)$. We lift $\overline{\gamma}$ to an action γ of $\mathbb{R}$ on B. Apply Proposition 2.4 to this action, and use Lemmas 2.3 and 2.5 to compute the K-theory of the resulting mapping cylinder. We get an exact sequence

$$
\begin{array}{ccccc}
RK_1(\mathscr{K}^\infty \otimes A) & \longrightarrow & RK_0(\mathscr{S}(\mathbb{R}, B, \gamma)) & \longrightarrow & RK_0(\mathscr{K}^\infty \otimes A) \\[2pt]
{\scriptstyle \mathrm{id} - (\lambda \otimes \alpha_{-1})_*}\big\uparrow & & & & \big\downarrow{\scriptstyle \mathrm{id} - (\lambda \otimes \alpha_{-1})_*} \\[2pt]
RK_1(\mathscr{K}^\infty \otimes A) & \longleftarrow & RK_1(\mathscr{S}(\mathbb{R}, B, \gamma)) & \longleftarrow & RK_0(\mathscr{K}^\infty \otimes A),
\end{array}
$$

where $\lambda \in \mathrm{Aut}(\mathscr{K}^\infty)$ is conjugation by the bilateral shift. Since λ is homotopic to the trivial automorphism of $\mathscr{K}^\infty$ (see [10, Lemma 2.7]), we can use stability in K-theory to delete it and $\mathscr{K}^\infty$. The smooth Thom isomorphism allows us to replace $RK_i(\mathscr{S}(\mathbb{R}, B, \gamma))$ by $RK_{1-i}(B)$. Rotating the exact sequence two spaces counter-clockwise then gives the exact sequence of the theorem, except for the identification of the maps ι_*.

The proof of this identification is not given in [1], so we give details here. We must show commutativity of the following diagram, with maps as described below:

$$
\begin{array}{ccccc}
RK_0(A) & \longrightarrow & RK_0(\mathscr{K}^\infty \otimes A) & \longrightarrow & RK_1(S_\infty(\mathscr{K}^\infty \otimes A, \beta)) \\[2pt]
{\scriptstyle \iota_*}\big\downarrow & & & & \big\downarrow{\scriptstyle \iota_*} \\[2pt]
RK_0(B) & \longrightarrow & RK_1(\mathscr{S}(\mathbb{R}, B, \gamma)) & \longrightarrow & RK_1(M_\infty(\mathscr{K}^\infty \otimes A, \beta)).
\end{array}
$$

The vertical maps are induced by the inclusions. The maps across the top are stability followed by Bott periodicity (Lemma 2.2). The maps across the

bottom are the Thom isomorphism (Theorem 1.2.7) and the map induced by the isomorphisms in Proposition 2.4 and Lemma 2.5.

We consider an arbitrary class in $RK_0(A)$. Replacing A by $M_2((\mathscr{K}^\infty \otimes A)^+)$, throughout, we may assume our class is represented by an idempotent $p \in A$. (Compare with the argument using the second diagram in the proof of Theorem 1.2.7.) We begin by computing the image of $[p]$ via the lower left corner.

Let q be the image of p in B. For the next step, choose $f_0 \in \mathscr{S}(\mathbb{R})$ whose Fourier transform $\hat{f}_0$ has support in $(0, 1)$ and such that $1 + \hat{f}_0$ is an invertible element of $\mathscr{S}(\mathbb{R})^+$ (for pointwise multiplication in $\mathscr{S}(\mathbb{R})$) which represents the standard generator of $RK_1(\mathscr{S}(\mathbb{R}))$. Regarding elements of $\mathscr{S}(\mathbb{R}, B, \gamma)$ as functions from $\mathbb{R} \times \mathbb{Z}$ to A, let $f \in \mathscr{S}(\mathbb{R}, B, \gamma)$ be $f(s, n) = \delta_{0n} f_0(s) p$, that is, $f = f_0 \otimes q$. Since q is γ-invariant, this formula actually yields an invertible element $1 + f$ in the unitization of each crossed product $\mathscr{S}(\mathbb{R}, B, \gamma^{(r)})$, where $\gamma_s^{(r)} = \gamma_{rs}$ for $r \in [0, 1]$. Definition 1.2.1 (of the Thom map) shows that $1 + f$ represents the image of $[q]$ in $RK_1(\mathscr{S}(\mathbb{R}, B, \gamma))$.

We now compute the image of f under the isomorphisms of Proposition 2.4 and Lemma 2.5. We first apply the map Ψ from the proof of Proposition 2.4, and then apply pointwise on $[0, 1]$ the maps $\mathscr{F}$ and Γ from the proof of Lemma 2.5. Writing the result as a function of $s \in [0, 1]$ and $l, n \in \mathbb{Z}$, combining exponentials, and rearranging slightly, we get

$$(\Gamma \circ \mathscr{F} \circ \Psi)(f)(s, l, n) = \sum_{k \in \mathbb{Z}} \int_0^1 \exp(2\pi i(tn + st + sk)) \alpha_{-l}(f(t + k, l - n)) \, dt.$$

Since nk is an integer, we can rewrite the exponential as $\exp(2\pi i(t+k)(s+n))$. Putting in the definition of f, we find that $(\Gamma \circ \mathscr{F} \circ \Psi)(f)(s, l, n) = 0$ for $l \neq n$, and

$$(\Gamma \circ \mathscr{F} \circ \Psi)(f)(s, n, n) = \alpha_{-n}(p) \sum_{k \in \mathbb{Z}} \int_0^1 \exp(2\pi i(t + k)(s + n)) f_0(t + k) \, dt$$

$$= \alpha_{-n}(p) \hat{f}_0(s + n).$$

Since $s \in [0, 1]$, we have $\hat{f}_0(s+n) = 0$ for $n \neq 0$. We conclude that the image of $[p]$ in $RK_1(M_\infty(\mathscr{K}^\infty \otimes A, \beta))$ is represented by $a = 1 + (\hat{f}_0|_{[0, 1]}) \otimes e_{00} \otimes p$, where $e_{00} \in \mathscr{K}^\infty$ is $e_{00}(l, n) = \delta_{0l} \delta_{0n}$.

We now go via the upper right corner. The image of $[p]$ in $RK_0(\mathscr{K}^\infty \otimes A)$ is $[e_{00} \otimes p]$. One easily checks that the image of this under Bott periodicity can be obtained as

$$[1 + (\hat{f}_0|_{[0, 1]}) \otimes e_{00} \otimes p] \in RK_1(S_\infty(\mathscr{K}^\infty \otimes A, \beta)),$$

by the choice of $\hat{f}_0$. But this is just a. So we have shown that the diagram commutes. □

We define the notion of an isometric action of $\mathbb{Z}$ on the Fréchet algebra just as we did for an action of $\mathbb{R}$ in Definition 1.3.3. Let

$$L_1(\mathbb{Z}, A, \alpha) = \left\{ f : \mathbb{Z} \to A \, \middle| \, \sum_{m \in \mathbb{Z}} \|f(m)\|_k < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

If α acts isometrically on A, then $L_1(\mathbb{Z}, A, \alpha)$ is a Fréchet algebra under convolution.

2.7 Lemma. *Let A be a Fréchet algebra with isometric action of $\mathbb{Z}$. Then the inclusion map $\mathscr{S}(\mathbb{Z}, A, \alpha) \hookrightarrow L_1(\mathbb{Z}, A, \alpha)$ is an isomorphism on K-theory.*

Proof. Let $\| \ \|_k$ be increasing submultiplicative α-isometric seminorms for A. Let A_k be the completion of $A/\operatorname{Ker}(\| \ \|_k)$ in $\| \ \|_k$, and let $B_k = L_1(\mathbb{Z}, A_k, \alpha)$. Then the inverse limit $\varprojlim B_k$ is equal to $L_1(\mathbb{Z}, A, \alpha)$. Define

$$\|f\|_{n,k} = \sum_{m \in \mathbb{Z}} (1 + |m|)^n \|f(m)\|_k, \qquad f \in \mathscr{S}(\mathbb{Z}, A, \alpha).$$

Let $B_{n,k}$ be the completion of $\mathscr{S}(\mathbb{Z}, A, \alpha)/\operatorname{Ker}(\| \ \|_{n,k})$ in $\| \ \|_{n,k}$. The seminorms $\| \ \|_{n,k}$ are submultiplicative under convolution since the action is isometric. Moreover, we have $\|f * g\|_{n,k} \le 2^n (\|f\|_{n,k} \|g\|_{0,k} + \|f\|_{0,k} \|g\|_{n,k})$. The proof is similar to, but simpler than, the computation in the proof of Theorem 1.3.4. Now finish just as there. □

2.8 Corollary. *If α is an isometric action of $\mathbb{Z}$ on a Fréchet algebra A, then we have the following natural six term exact sequence in K-theory.*

$$
\begin{array}{ccccc}
RK_0(A) & \xrightarrow{\mathrm{id} - (\alpha_{-1})_*} & RK_0(A) & \xrightarrow{\iota_*} & RK_0(L_1(\mathbb{Z}, A, \alpha)) \\
\partial \uparrow & & & & \downarrow \partial \\
RK_1(L_1(\mathbb{Z}, A, \alpha)) & \xleftarrow{\iota_*} & RK_1(A) & \xleftarrow{\mathrm{id} - (\alpha_{-1})_*} & RK_1(A).
\end{array}
$$

If A is a Banach algebra, then so is $L_1(\mathbb{Z}, A, \alpha)$, and we may replace RK by K everywhere in the diagram.

Proof. This follows from Theorem 2.6 and Lemma 2.7. For the statement about Banach algebras, see [10, Corollary 7.8]. □

3. Examples and applications

We give several examples and applications of our theorems.

3.1 Example. For $G = \mathbb{Z}$ or $\mathbb{R}$, there are many examples of actions α on locally compact spaces M which induce an m-tempered action on a suitable Fréchet algebra $\mathscr{S}(M)$ of "smooth" functions on M. (See [13, Examples 7.20, 6.26–6.27, 2.6–2.7] and [12, Example 5.18]. The examples [12, Examples 5.19, 5.23–5.24, 5.26–5.27] also include examples of m-tempered actions, along with many actions that are *not* m-tempered with respect to the absolute value function—see Definition 1.1.3 and [12, Definition 3.1.1].) Our smooth Thom isomorphism and Pimsner-Voiculescu exact sequence apply to these examples. We note, however, that the K-theory of the smooth crossed product can often be computed directly from Lemma 1.1.9(1) and the C^*-algebras results, because $\mathscr{S}(G, \mathscr{S}(M), \alpha)$ is often spectral invariant in $C^*(G, C_0(M), \alpha)$. (See [13, Corollary 7.16].)

Two specific examples are the canonical smooth subalgebra $\mathscr{S}(\mathbb{Z}, C^\infty(\mathbb{T}), \alpha)$ of the (rational or irrational) rotation algebra, and $\mathscr{S}(\mathbb{R}, \mathscr{S}(H/K), \alpha)$, where H is the unipotent 3×3 upper triangular real matrices, $\mathbb{R}$ the subgroup corresponding to the first row and second column, and K any closed subgroup of H.

3.2 **Example.** Let $G = \mathbb{Z}$ or $\mathbb{R}$, and let $\varphi\colon A \to B$ be a homomorphism which is equivariant for the m-tempered actions α and β on the Fréchet algebras A and B. Assume that $\varphi_*\colon RK_*(A) \to RK_*(B)$ is an isomorphism. Then $RK_*(\mathscr{S}(G, A, \alpha)) \to RK_*(\mathscr{S}(G, B, \beta))$ is also an isomorphism. If β is isometric and B is a Banach algebra, then $RK_*(\mathscr{S}(G, A, \alpha)) \to K_*(L_1(G, B, \beta))$ is an isomorphism, and if B is a C^*-algebra, then $RK_*(\mathscr{S}(G, A, \alpha)) \to K_*(C^*(G, B, \beta))$ is an isomorphism. For $G = \mathbb{R}$, these statements are immediate from the Thom isomorphisms for the various crossed products, and for $G = \mathbb{Z}$ they follow from the Pimsner-Voiculescu exact sequences via the Five Lemma.

This argument applies without any assumption on φ beyond that it is a continuous equivariant homomorphism which is an isomorphism on K-theory. But we are of course most interested in the case where φ is the inclusion of a dense subalgebra A of a Banach or C^*-algebras B, with A Fréchet in its own topology. If A is spectral invariant in B, then φ is automatically an isomorphism on K-theory, by Lemma 1.1.9(1). It follows that the inclusion $\mathscr{S}(G, A, \alpha) \to L_1(G, B, \beta)$ (or $\mathscr{S}(G, A, \alpha) \to C^*(G, B, \beta)$) is an isomorphism on K-theory. Cases in which the smooth crossed product is spectral invariant in the C^* crossed product are mentioned in Example 3.1, but here the point is that one gets the isomorphism of the K-theory of the crossed products without knowing anything about spectral invariance of the crossed products.

For example, consider Theorem 12.5 of [9], and assume that the action is m-tempered, so that Nest's crossed product is the same as ours. Using RK_* in place of Nest's K_*^p for the dense subalgebras, we get that if the inclusion of the dense subalgebra is an isomorphism on K-theory before taking crossed products, then it is again an isomorphism (not just surjective) after taking crossed products.

The discussion above generalizes to a closed subgroup G of a connected, simply connected, nilpotent Lie group H. Such a group G is a semidirect product $\mathbb{Z} \rtimes \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z} \rtimes \mathbb{R} \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$. Let α and β be an m-tempered action of G on A and B for which φ is equivariant. (Here m-temperedness is meant with respect to the gauge from [12, Theorem 1.5.13]. See also [12, Example 1.5.14 and 12, Definition 3.1.1].) Then, decomposing the smooth crossed products $\mathscr{S}(G, A, \alpha)$ and $\mathscr{S}(G, B, \beta)$ (defined in [12, §2]) into successive smooth crossed products by $\mathbb{R}$ and then by $\mathbb{Z}$, we see by iterating the above results that $\mathscr{S}(G, A, \alpha) \to \mathscr{S}(G, B, \beta)$ is an isomorphism on K-theory.

Consider the special case that B is a C^*-algebra, and β is an action via $*$-automorphisms. Then the inclusion map $\mathscr{S}(G, B, \beta) \hookrightarrow C^*(G, B, \beta)$ is spectral invariant with dense image by [13, Corollary 7.16]. (The same result in the special case $G = \mathbb{Z}$ is also done in [2, Theorem 2.3.3].) Hence $\mathscr{S}(G, A, \alpha) \to C^*(G, B, \beta)$ is an isomorphism on K-theory.

3.3 **Example.** Let U be the annulus $\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 2\}$. Let $A(U)$ be the Banach algebra of continuous functions from $\overline{U}$ to $\mathbb{C}$ which are holomorphic on U. Let $C(U)$ be the Fréchet algebra of all continuous functions on U, with the topology of uniform convergence on compact subsets, and let $H(U)$ be the closed subalgebra of holomorphic functions on U. Then the rotation by an irrational angle θ defines an isometric action on each of these algebras. The maximal ideal spaces of these algebras are either $\overline{U}$ or U, and the inclusion

of the unit circle $\mathbb{T}$ into each of these spaces is a homotopy equivalence. Using Theorem 7.15 of [10], it is easy to see that the restriction map from any of these algebras to $C(\mathbb{T})$ is an isomorphism on K-theory. It follows from the previous example that $L_1(\mathbb{Z}, A(U), \theta)$, $L_1(\mathbb{Z}, C(U), \theta)$, and $L_1(\mathbb{Z}, H(U), \theta)$ all have the same K-theory as the irrational rotation C^*-algebra A_θ. The same holds for the smooth crossed products $\mathscr{S}(\mathbb{Z}, A(U), \theta)$, etc.

Let A be the Banach algebra of continuous functions on $\mathbb{T}$ with absolutely convergent Fourier series, with the norm $\|f\| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$. A similar argument to the above also shows that $L_1(\mathbb{Z}, A, \theta) \to A_\theta$ is an isomorphism on K-theory.

In place of U above, one can also use $\mathbb{C}$, the open unit disk D, or $\mathbb{C} - \overline{D}$. The K-theory of the crossed product is always the same as that of $C^*(\mathbb{Z})$ for $\mathbb{C}$ and for D, and the same as that of A_θ for $\mathbb{C} - \overline{D}$. One can also let $\mathbb{R}$ act on all these algebras via rotation, with similar results, or one can let $\mathbb{R}$ act by translation on similar algebras gotten by using a half plane or strip in place of a disk or an annulus.

In some of the cases above, the K-theory of the crossed product can be computed by other methods. For example, the disk algebra $A(D)$ is equivariantly homotopy equivalent to $\mathbb{C}$, via evaluation at 0. Therefore, $L_1(\mathbb{Z}, A(D), \theta)$ is homotopy equivalent of $L_1(\mathbb{Z})$. Also, $\mathscr{S}(\mathbb{Z}, C^\infty(\mathbb{T}), \theta)$ is dense and spectral invariant in both $L_1(\mathbb{Z}, A, \theta)$ and A_θ [13, Theorem 6.7, Corollary 7.16], so that $L_1(\mathbb{Z}, A, \theta) \to A_\theta$ is an isomorphism on K-theory by Lemma 1.1.9(1). However, the methods vary from case to case, and it is not obvious that they apply to all the cases above.

For other examples, consider the irrational rotation on $H^\infty(D)$, $H^\infty(U)$, or the measure algebra $M(\mathbb{T})$. We will not attempt to compute the K-theory of these crossed products. (Note that the action of $\mathbb{R}$ on these algebras is not continuous.)

3.4 Example. We give an example of a smooth crossed product of $\mathbb{R}$ with A in which A is a subalgebra of a Banach algebra but not spectrally invariant in any Banach algebra. The algebra in this example is thus essentially different from the algebras of unbounded functions in Example 3.3. For $n \in \mathbb{N}$, define

$$A_n = \left\{ f \in L_1(\mathbb{R}) \,\middle|\, \int_{\mathbb{R}} e^{n|r|} |f(r)| dr < \infty \right\} \quad \text{and} \quad A = \bigcap_{n \in \mathbb{N}} A_n = \varprojlim A_n.$$

Then each A_n is a Banach algebra under convolution and A is a Fréchet algebra. Let $\mathbb{R}$ act on A via $\alpha_s(f)(r) = e^{isr} f(r)$. This action is isometric and hence m-tempered. We will compute the K-theory of the smooth crossed product, a somewhat strange subalgebra of $\mathscr{K}^\infty$, by computing the K-theory of A and applying the Thom isomorphism. (It is possible to compute the K-theory of this crossed product more directly, but the computation is more complicated than for A.)

We compute $RK_*(A)$ using Theorem 7.15 of [10], which requires that we know the maximal ideal space $\text{Max}(A^+)$ of closed maximal ideals and its compactly generated topology. Every complex number z defines a character $\eta_z(f) = \int_{\mathbb{R}} e^{izr} f(r) dr$ of A (with obvious extension to a character of the unitization A^+). Note that A^+ also has the character $\eta_\infty(\lambda 1 + f) = \lambda$ which vanishes on A. We show that these are in fact all the characters of A^+. Let ξ

be any character of A^+. If ξ vanishes on A, then clearly $\xi = \eta_\infty$. Otherwise, ξ is a continuous linear map from A to the Banach space $\mathbb{C}$. Therefore there must be some n such that ξ lifts to a continuous linear map from A_n to $\mathbb{C}$. It follows that there is $\chi_0 \in L_\infty(\mathbb{R})$ such that $\xi(f) = \int_\mathbb{R} f(r)e^{n|r|}\chi_0(r)\,dr$. Using $\xi(f*g) = \xi(f)\xi(g)$, a standard argument shows that $e^{n|r|}\chi_0(r)$ coincides almost everywhere with a continuous function χ which satisfies $\chi(r+s) = \chi(r)\chi(s)$. (See, for example, [5, proof of Theorem VII.9.6].) It follows that $\chi(r) = e^{izr}$ for some complex number z. Since $|\chi(r)| \leq \|\chi_0\|_\infty e^{n|r|}$, we get $|\mathrm{Im}(z)| \leq n$. This shows that $\xi = \eta_z$ with $|\mathrm{Im}(z)| \leq n$. Thus, the maximal ideal space $\mathrm{Max}(A^+)$ is exactly $\mathbb{C} \cup \{\infty\}$.

We further see from this argument that $\mathrm{Max}(A_n^+)$ is the set

$$K_n = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \leq n\} \cup \{\infty\},$$

topologized as the one point compactification of the strip. (We check that this is the right topology. By compactness and metrizability, it suffices to show that if $z_k \to z$, then $\eta_{z_k} \to \eta_z$. For $z \neq \infty$, this follows from the Lebesgue Dominated Convergence Theorem. If $z = \infty$, we use the following argument. For $s \in [-n, n]$ define $f_s(r) = e^{-sr}f(r)$ for $f \in A_n$. Then $s \mapsto \hat{f}_s$ is a continuous map from $[-n, n]$ to $C_0(\mathbb{R})$ and so has compact image. Hence for each $\varepsilon > 0$ the image is contained in one of the open sets $U_{\varepsilon,m} = \{\xi \in C_0(\mathbb{R}) \mid |\xi(r)| < \varepsilon$ for all $r \notin [-m, m]\}$. Hence if $|\mathrm{Re}(z_k)| > m$, then $|\hat{f}(z_k)| = |\hat{f}_{\mathrm{Im}(z_k)}(\mathrm{Re}(z_k))| < \varepsilon$. So $z_k \to \infty$ implies that $\eta_{z_k}(f) = \hat{f}(z_k) \to 0 = \eta_\infty(f)$.) It follows as in the proof of Lemma 7.14 of [10] that the compactly generated topology on $\mathrm{Max}(A^+)$ is the topology it gets from the identification with $\varinjlim K_n$ (not the one point compactification of $\mathbb{C}$).

By [10, Theorem 7.15], $RK_*(A) = RK_*(C_A)$, where C_A is the closed subspace of $C(\mathrm{Max}(A^+))$ of functions which vanish at ∞. It is easily seen, using the previous paragraph, that the restriction map $C_A \to C_0(\mathbb{R})$ is a homotopy equivalence of Fréchet algebras. It follows that $A \to C_0(\mathbb{R})$ is an isomorphism on K-theory. Taking crossed products by $\mathbb{R}$ and using the Thom isomorphism, we find that $RK_*(\mathscr{S}(\mathbb{R}, A, \alpha)) \to K_*(\mathscr{K})$ is an isomorphism.

We remark that $\exp(-r^2)$ is an element of A which has noncompact spectrum (namely $\mathbb{C}$). Therefore the invertible group A^+ is not open. It follows that A cannot be spectral invariant in any Banach algebra.

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